# Fokker-Planck collision model and Monte Carlo method

The first part of the derivation follows closely the book *Collisional transport in magnetized plasmas* of Helander, Sigmar (2002), pp.22-24. Extensive information on the topic of kinetic theory and its various applications can be found in the book *Stochastic processes in physics and chemistry* of N.G. Van Kampen (2007).

We look at collisional effects to the velocity dependency of the distribution function $f(v,t)$. Since no particles are created or destroyed by collisions, after a sufficiently short time-step $\Delta t$ the change in $f$ at a fixed point $v$ of velocity space can be written as

$$f(v,t+\Delta t) = \int \mathrm{d}\Delta v\, f(v-\Delta v,t) F(v-\Delta v, \Delta v). \tag{1}$$

The function $F(v-\Delta v,\Delta v)$ quantifies the probability with which a particle of velocity $v-\Delta v$ will be scattered to have velocity $v$, and $f(v-\Delta v,t)$ measures the population at the velocity space point from which scattering occurs. Integration corresponds to a summation over all possible distances $\Delta v$ from $v$. The central assumption to obtain a Fokker-Planck collision operator is the sufficiently fast decay of $F$ with $\Delta v$. Physically this means that most of the velocity changes by a single collision are small, which is a good approximation for Coulomb collisions in plasmas or for Brownian motion in chemistry (Kramers' equation). An expansion up to second order around $v$ in the first argument $v-\Delta v$ yields

$$\begin{aligned} f(v,t+\Delta t) \approx \int \mathrm{d}\Delta v\, \Bigg( & f(v,t)F(v,\Delta v) - \Delta v \frac{\partial (f(v,t)F(v,\Delta v))}{\partial v} \\ & + \frac{\Delta v^2}{2}\frac{\partial^2}{\partial v^2}(f(v,t)F(v,\Delta v)) \Bigg). \end{aligned} \tag{2}$$

Since no particles can be lost, the integral over $F(v,\Delta v)$ across the whole range of $\Delta v$ has to fulfil the normalisation condition

$$\int \Delta v\, F(v,\Delta v) = 1, \tag{3}$$

which means that the first term can be integrated to yield just $f(v,t)$. Moving this term to the left-hand side and division by $\Delta t$ yields the partial time derivative of $f$ due to collisions in the limit $\Delta t \to 0$ with

$$\begin{aligned} \left(\frac{\partial f}{\partial t}\right)_c &= \lim_{\Delta t\to 0} \frac{f(v,t+\Delta t) - f(v,t)}{\Delta t} \\ &\approx -\frac{\partial}{\partial v}\left( \underbrace{\frac{\langle \Delta v\rangle}{\Delta t}}_{A(v)} f(v,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left( \underbrace{\frac{\langle \Delta v^2\rangle}{\Delta t}}_{B(v)} f(v,t)\right), \end{aligned} \tag{4}$$

where we have defined

$$A(v) \equiv \frac{\langle \Delta v\rangle}{\Delta t}, \quad B(v) \equiv \frac{\langle \Delta v^2\rangle}{\Delta t}$$

containing the integral over $\mathrm{d}\Delta v$ as moment averages for powers of $\Delta v$ with weight $F$,

$$\langle \Delta v^k \rangle = \int \mathrm{d}\Delta v\, F(v,\Delta v)\Delta v^k. \tag{5}$$

If we furthermore assume $A$ to be linear in $v$ and $B$ constant, we can write this collision term in the so-called Ornstein-Uhlenbeck form

$$\left(\frac{\partial f}{\partial t}\right)_c = \hat{L}_c f = \nu_c \frac{\partial}{\partial v}\left( v\, f + v_T^2 \frac{\partial f}{\partial v}\right), \tag{6}$$

where we have expressed Fokker-Planck coefficients $A$ and $B$ via collision frequency $\nu_c$ and thermal velocity $v_T$ to fulfil

$$A \equiv -v\nu_c, \tag{7}$$

$$B \equiv 2\nu_c v_T^2, \tag{8}$$

thus uniquely defining first and second moments $\langle \Delta v \rangle$, $\langle \Delta v^2 \rangle$ of $\Delta v$ with respect to $F$. The choice of the name *thermal velocity* for $v_T$ stems from the convergence of $f$ towards a stationary thermalised state $f_\infty$ given by a Gaussian in $v$ (Boltzmann distribution in $E = \frac{mv^2}{2}$ for 1D or Maxwellian for 3D) at $t \to \infty$ with

$$f_\infty(v) = \frac{1}{\sqrt{2\pi v_T^2}} e^{-\frac{v^2}{2v_T^2}}. \tag{9}$$

It can be easily checked that $\hat{L}_c f_\infty = 0$, so a characterisation for the stationary $f_\infty$ is being an eigenfunction of $\hat{L}_c$ with eigenvalue 0.

To define a random process for a Monte Carlo method representing the Ornstein-Uhlenbeck operator, we require a random distribution of $\Delta v$ whose first and second moment are given by Eqs. (7-8). Effectively this means to sample from a model for the distribution function $F(\Delta v)$. A straightforward choice is a normal (Gaussian) distribution

$$F(\Delta v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(\Delta v - \mu)^2}{2\sigma^2}} \tag{10}$$

with mean

$$\mu = \langle \Delta v \rangle = -\nu_c v \Delta t \tag{11}$$

and variance

$$\sigma^2 = \langle \Delta v^2 \rangle = 2\nu_c v_T^2 \Delta t. \tag{12}$$

In each time-step we draw a random sample from this distribution by

$$\Delta v = -\nu_c v \Delta t + \sqrt{2\nu_c v_T^2}\Theta\sqrt{\Delta t}, \tag{13}$$

where $\Theta$ is a random number sampled from the standard normal distribution. As an alternative to using the normal distribution as a model for $F$, it would also be possible to use equally uniformly distributed random numbers centred around $\mu$ and with the correct scaling of the variance.