Kinetic Theory in Plasma Physics
Lecture notes

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Contents

Preface

This set of lecture notes is intended as the basis of the lecture as well as study material for students. The focus will be the application of the methods to fusion plasmas. Note that there are questions at the end of each chapter. These questions act as a guideline for preparing for the exam. For the best learning effect, try to answer them without looking at the lecture notes. Only when no solution comes to mind, read the notes and focus on what you missed.

Equations or formulas that are important are written in boxes. It would be beneficial to understand or even memorize these as they frequently occur when in the context of plasma physics.

Units

As is often used in fusion physics, we employ the cgs-unit system.

$$\text{Magnetic field } [B] = \mathrm{G} \tag{1}$$
$$\text{Length } [r] = \mathrm{cm} \tag{2}$$
$$\text{Electric field } [E] = \mathrm{statV\ cm^{-1}} \tag{3}$$
$$\text{Mass } [m] = \mathrm{g} \tag{4}$$
$$\text{Time } [t] = \mathrm{s} \tag{5}$$
$$\text{Velocity } [v] = \mathrm{cm\ /\ s} \tag{6}$$
$$\text{Particle density } [n] = \mathrm{cm^{-3}} \tag{7}$$
$$\ldots$$

To go from SI units to cgs units, use the following prescription:

$$\varepsilon_0 \to \frac{1}{4\pi} \tag{8}$$
$$\mu_0 \to \frac{4\pi}{c} \tag{9}$$
$$\mathbf{B} \to \frac{1}{c}\mathbf{B} \tag{10}$$
$$\mathbf{A} \to \frac{1}{c}\mathbf{A} \tag{11}$$

Chapter 1

Primer on Plasma Physics

1.1 Introduction

A plasma is a partially or fully ionized gas that exhibits **collective behavior** due to Coulomb interactions. The property of collective phenomena is where the complexity of plasmas occurs. In principle, multiple "pictures" or models are used to investigate plasma under various circumstances. They vary in their complexity, but also in their validity range and the range of covered physical phenomena. Hence, not every model can be used to describe a certain situation. Also, using an unnecessarily complex model to describe a simple system is obviously undesirable.

As illustrated in figure 1.1, at the most fundamental level, a plasma can be described by the **particle picture**. This microscopic description follows each individual charged particle of the plasma. The motion is determined by the Lorentz force. As we will see, many properties of the particle picture carry on to the kinetic picture and, therefore, we will discuss the particle picture in more detail in the beginning of the lecture in section 1.2.

On the other end of the range is the **fluid picture**. Herein, plasma is described by macroscopic quantities like the density, pressure, averaged velocity, temperature and so on that vary in space and time. The fluid picture is derived from the kinetic picture. This is described in section 2.4.

In between particles and fluids is the statistical approach of the **kinetic picture**, where plasma is described by the distribution function of particles in phase space. That is, the distribution function tells us how many particles there are in a certain volume of phase space at some time. The main part of this lecture will be dedicated to the kinetic description and the equation that governs the evolution of the distribution function, the Vlasov equation.

Why are we interested in the kinetic picture? It is less precise than the particle picture and more conceptually complex than the fluid picture. However, firstly, it is (currently?) impossible to track the trajectories of the 10^{20}-ish

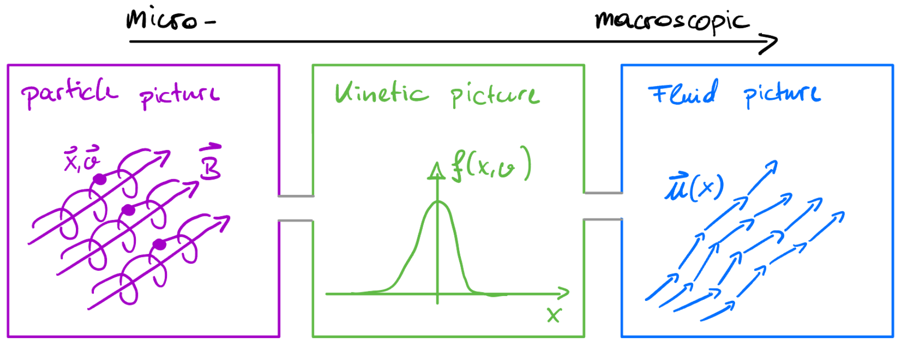

Figure 1.1: The different pictures of how to describe a plasma.

particles in a (fusion) plasma. Kinetic theory gives us a statistical description of this large amount of particles. Second, the fluid picture inherently assumes the velocity distribution of the plasma to be in equilibrium, which is rarely the case in a plasma. As such, the fluid picture omits physics like wave-particle interactions, instabilities, and turbulence.

Kinetic theory comes in multiple flavors with various assumptions for specific situations. Most prominently, **drift-kinetic theory** is used to describe **transport phenomena** in fusion plasmas [5], and **gyro-kinetic theory** is used to describe **turbulence**. However, in this lecture, we will focus on the full kinetic theory, and will only briefly touch on drift-kinetics and gyro-kinetics if there is time.

1.2 Particle picture

Before going into kinetic theory, we shall recall facts and ideas about the particle picture, specifically the motion of particles in electromagnetic fields. A particle in electromagnetic fields is subject to the Lorentz force, where the governing equations of motion are

Single-particle equations of motion

$$\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \mathbf{v} \tag{1.1}$$

$$\frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \frac{Ze}{m}\Big(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\Big), \tag{1.2}$$

where Z is the charge number, e is the elementary charge and m is the mass of the particle. For a first example, let's consider the motion of a particle in a constant magnetic field $\mathbf{B} = B\mathbf{e}_z$ without an electric field. In this case, the force equation gives

$$\dot{\mathbf{v}} = \frac{Ze}{mc}\mathbf{v} \times \mathbf{B} \tag{1.3}$$
$$= \omega_c \mathbf{v}_\perp, \tag{1.4}$$

where we have the

Cyclotron frequency

$$\omega_c = \frac{ZeB}{mc} \tag{1.5}$$

or gyro frequency which is often also written with Ω, and $\mathbf{v}_\perp = \mathbf{v} \times \mathbf{h} = (v_y, -v_x, 0)$ is the velocity perpendicular to the magnetic field. Here, we also defined the a general notation for the direction of the magnetic field, $\mathbf{h} = \mathbf{B}/B$. We can also explicitely write the equations,

$$\dot{v}_x = \omega_c v_y, \qquad \ddot{r}_x = \omega_c \dot{r}_y, \tag{1.6}$$
$$\dot{v}_y = -\omega_c v_x, \qquad \ddot{r}_y = -\omega_c \dot{r}_x, \tag{1.7}$$
$$\dot{v}_z = 0. \qquad \ddot{r}_z = 0. \tag{1.8}$$

Let's solve the equations for the velocity. Deriving the first equation with respect to time and substituting the second equation, we have

$$\ddot{v}_x + \omega_c^2 v_x = 0, \tag{1.9}$$

which is solved by either a sine or a cosine function. However, considering also the second equation $\dot{v}_y = -\omega_c v_x$, it can only be a sine function to arrive at the right sign. Therefore, the solution is

$$v_x = v_\perp \sin(\omega_c t + \phi_0) + v_{x0}, \tag{1.10}$$
$$v_y = v_\perp \cos(\omega_c t + \phi_0) + v_{y0}. \tag{1.11}$$

Hence, the velocity vector is given by

$$\mathbf{v}(t) = v_\parallel \mathbf{h} + \mathbf{v}_\perp \tag{1.12}$$
$$= v_{z0}\mathbf{h} + v_\perp \Big(\cos(\omega_c t)\mathbf{e}_x - \sin(\omega_c t)\mathbf{e}_y\Big), \tag{1.13}$$

where we have set the initial velocity in x and y, and the initial phase ϕ_0 to zero for brevity. So, the magnetic field does not accelerate the particle along its direction. However, perpendicular to the magnetic field, the particle is constantly accelerated in a circular motion.

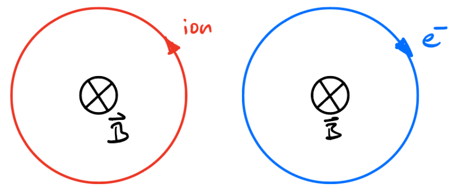

Figure 1.2: The direction of the gyro motion of (positively charged) ions and electrons in a magnetic field.

Next, we solve for the particle position by integrating the velocity

$$\mathbf{r}(t) = \int_{t_0}^{t} \mathrm{d}t' \ \mathbf{v}(t') \tag{1.14}$$

$$= v_{z0} t \mathbf{h} + \frac{v_\perp}{\omega_c}\Big(\sin(\omega_c t)\mathbf{e}_x + \cos(\omega_c t)\mathbf{e}_y\Big), \tag{1.15}$$

where we set $t_0 = 0$ and assumed zero initial conditions, $\mathbf{r}_0 = 0$. We see that the particle orbit in a constant magnetic field is comprised of two parts: a **linear motion along** the magnetic field and a **circulating motion perpendicular** to it. The circulating motion, commonly called gyration, gyro motion, cyclotron motion, etc., has a frequency $\omega_c = ZeB/(mc)$ that depends on the magnetic field and the parameters of the particle. Note that the frequency is signed, i.e. the direction of rotation depends on the sign of the charge. For ions ($Z > 0$), the rotation is clockwise in the $\mathbf{e}_x$, $\mathbf{e}_y$ plane, while for electrons it is counter-clockwise as sketched in figure 1.2 Note that the gyration direction is independent of the sign of the parallel velocity, but, of course, depends on the sign of the magnetic field.

Length scale: The radius of the gyration is called the **Larmor radius** or gyro radius,

Larmor radius

$$\rho_L = \frac{v_\perp}{\omega_c}. \tag{1.16}$$

This is an important length occurring in magnetically confined plasmas and often serves as a small parameter (in comparison to the system size or the magnetic field scale length) for Taylor expansions. We will encounter such an expansion for example in section 3.1. Further, the Larmor radius depends on the mass of the considered particle. In particular, for electrons, the Larmor radius is significantly smaller as for ions (tens of microns versus a few millimeter). This can be exploited in modeling of fusion plasmas, specifically, by ignoring the

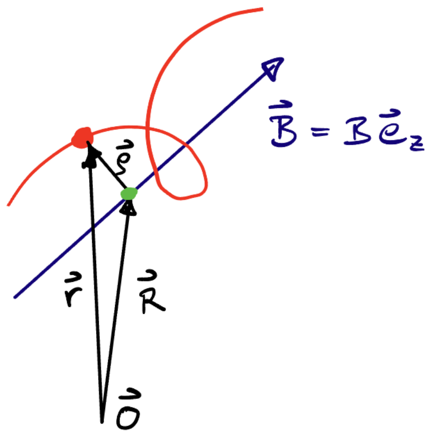

Figure 1.3: Particle motion dissected into guiding center (green dot) motion and particle (red dot) motion.

gyro motion of electrons and consider its guiding center position instead of the actual particle position. This is done in drift-kinetic theory (c.f. section 3.1), for example.

Time scale: The gyro motion not only defines a length scale, but also a time scale with the cyclotron frequency, $\tau_c = 2\pi/\omega_c$. The cyclotron frequency is rather high, e.g. $\sim 10^{11}$ Hz for electrons in typical fusion plasma conditions (B in the order of a few tesla). For ions, which are much heavier, the frequency is accordingly lower, e.g. a factor for hydrogen $m_p/m_e \approx 2000$ or deuterium $m_D/m_e \approx 4000$ lower[1].

Note that in an inhomogeneous magnetic field, both the Larmor radius and the cyclotron frequency will vary with the position of the particle.

Often, the particle position is split into

$$\mathbf{r}(t) = \mathbf{R}(t) + \boldsymbol{\rho}(t), \tag{1.17}$$

i.e. into the slow guiding center motion

$$\mathbf{R}(t) = (r_{z0} + v_{\parallel} t)\mathbf{h} + r_{x0}\mathbf{e}_x + r_{y0}\mathbf{e}_y, \tag{1.18}$$

and the fast gyro motion

$$\boldsymbol{\rho}(t) = \rho_L \Big(\sin(\omega_c t + \phi_0)\mathbf{e}_x + \cos(\omega_c t + \phi_0)\mathbf{e}_y \Big). \tag{1.19}$$

A sketch of this split can be seen in figure 1.3.

[1] These mass ratios are very often useful for order of magnitude estimations. So, keep them in mind.

1.2.1 Adding an electric field

So far, we have focused on the motion of a charged particle in a constant magnetic field. What happens if we add an electric field? Consider Lorentz' force law

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = Ze\mathbf{E} + \frac{Ze}{c}\mathbf{v}\times\mathbf{B}. \tag{1.20}$$

Parallel to the magnetic field, we have

$$m\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = ZeE_\parallel, \tag{1.21}$$

that is, an electric field parallel to the magnetic field freely accelerates the particle. On the other hand, perpendicular to the magnetic field, we have the equation

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\times\mathbf{B} = Ze\mathbf{E}\times\mathbf{B} + \frac{Ze}{c}(\mathbf{v}\times\mathbf{B})\times\mathbf{B} \tag{1.22}$$

$$= Ze\mathbf{E}\times\mathbf{B} + \frac{Ze}{c}\left(\mathbf{B}(\mathbf{v}\cdot\mathbf{B}) - \mathbf{v}B^2\right) \tag{1.23}$$

$$= Ze\mathbf{E}\times\mathbf{B} + \frac{Ze}{c}\left(\mathbf{h}v_\parallel B^2 - \mathbf{v}B^2\right) \tag{1.24}$$

$$= Ze\mathbf{E}\times\mathbf{B} - \frac{Ze}{c}B^2\mathbf{v}_\perp, \tag{1.25}$$

where we have used the BAC-CAB rule and $\mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel\mathbf{h}$. From earlier we know that the $\mathbf{B}$ field leads to a fast circular motion of the particle in $\mathbf{v}_\perp$. This is contained in the term on the left hand side and the second term on the right hand side. However, there is a term appearing with the electric field. This term leads to a slow drift motion of the particle. Omitting the fast gyromotion contribution, the governing equation is

$$0 = Ze\mathbf{E}\times\mathbf{B} - \frac{Ze}{c}B^2\mathbf{v}_E \tag{1.26}$$

$E\times B$ drift velocity

$$\Rightarrow \mathbf{v}_E = \frac{c\mathbf{E}\times\mathbf{B}}{B^2}. \tag{1.27}$$

This is the so-called $E\times B$ drift velociiy. It is perpendicular to both the electric and magnetic field. A sketch of the $E\times B$ drift is shown in figure 1.4. We can understand the drift motion as follows: Without the electric field, the particle gyrates around magnetic field lines. When an electric field is added, the circular motion is accelerated in parts of the orbit and decelerated in others. Considering the Larmor radius $\rho_L = v_\perp/\omega_c$, this change in $v_\perp$ means that the Larmor radius is decreasing and increasing in different parts of the orbit. Hence, the particle drifts.

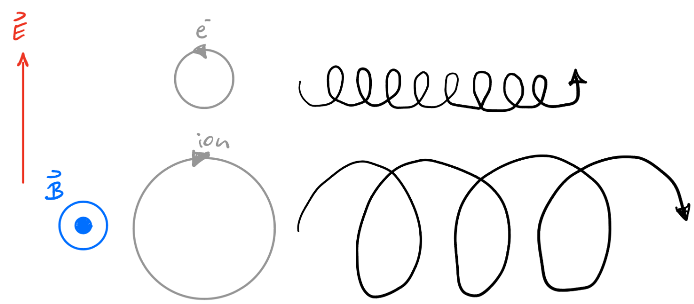

Figure 1.4: A beautiful sketch of the $E \times B$ drift motion.

It is important to note that the $E \times B$ drift is independent of the charge and mass of the particle. This means that the drift is the same for all particles in a plasma. Such a type of drift is called **ambipolar**. Since both positive and negative charge carriers drift with the same speed, there is no net electric field or current generated.

To summarize, a particle in a constant electric and magnetic field has three components to its motion

(1) A parallel motion that is accelerated by the electric field.

(2) A fast circular motion perpendicular to the magnetic field.

(3) A slow drift motion perpendicular to both the electric and magnetic field.

In general, when particles are subject to a force $\mathbf{F}$ they will drift in a magnetic field according to

Perpendicular drift velocity due to a force

$$\mathbf{v}_d = \frac{c}{Ze}\frac{\mathbf{F} \times \mathbf{B}}{B^2}. \tag{1.28}$$

For example, it is unavoidable on Earth, or anywhere else in the Universe for that matter, to have a gravitational force acting on the particles. This leads to a **gravitational drift** of the particles in a magnetic field. However, the gravitational drift is usually negligible in comparison to other drifts in a plasma.

Note that this drift velocity is perpendicular to the magnetic field. Of course, forces can act in the parallel as well.

1.2.2 Drifts in non-uniform magnetic fields

Generalising the particle motion to non-uniform electromagnetic fields can conceptually be thought of as the addition of drifts due to forces on the particles.

But, in general non-uniform electromagnetic fields greatly increase the complexity of the particle motion. Here, we will only shortly introduce the different drifts that can occur in such fields. A more detailed discussion can be found in the majority of standard plasma physics texts, e.g. [4].

Before going into the drifts, let's introduce the notion of the **magnetic dipole moment**. A magnetic dipole is generated for example by a permanent magnet, which has two poles: a north and a south pole. But, such a dipole field can also be generated by a current loop. The magnetic dipole moment μ is defined as

Magnetic dipole moment

$$\boldsymbol{\mu} = I\mathbf{A}. \tag{1.29}$$

Here, $\boldsymbol{\mu}$ is the magnetic moment which quantifies the direction and strength of the dipole field, I is the current flowing through the current loop and $\mathbf{A}$ is the area vector of the loop-enclosed area.

Considering the gyromotion of particles around magnetic field lines, they pose a similar scenario: the flow of a current around an area. In this case, the current is given by

$$I = \frac{eZ}{\tau_c}, \tag{1.30}$$

where $\tau_c = 2\pi/\omega_c$ is the cyclotron period. The area, on the other hand, is given by

$$A = \pi\rho_L^2. \tag{1.31}$$

Hence, the strength of the magnetic moment of a gyrating particle is given by

$$\mu = IA \tag{1.32}$$

$$= \frac{eZ}{\tau_c}\pi\rho_L^2 \tag{1.33}$$

$$= \frac{v_\perp^2}{\omega_c^2}\frac{eZ}{2\pi}\omega_c \tag{1.34}$$

Magnetic moment

$$\mu = \frac{mv_\perp^2}{2B}. \tag{1.35}$$

A sketch of the magnetic moment is shown in figure 1.5. The direction of the magnetic moment is such that it opposes the external field. Hence, gyrating particles in a plasma have a tendency to reduce the total magnetic field and therefore, plasma is a diamagnetic medium. It prevents external fields from entering.

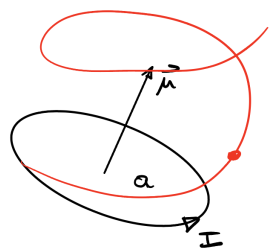

Figure 1.5: Magnetic moment of a charged particle gyrating in a magnetic field.

Hence, a gyrating particle in a magnetic field can be seen as a tiny magnet moving in an external magnetic field. This conceptual ansatz is also called **Pauli particle**.

Note that this is the perpendicular kinetic energy of the particle divided by the magnetic field strength, or

$$W_\perp = \frac{mv_\perp^2}{2} = \mu B. \tag{1.36}$$

Since the fast gyromotion is now encoded in the magnetic moment, this energy can be seen as a potential energy of the tiny magnet instead of the kinetic energy of the particle. From this potential, we can actually derive a force, if the magnetic field strength varies in space. The force is given by

$$\mathbf{F}_{\nabla B} = -\nabla(\mu B) = -\mu \nabla B - B \nabla \mu. \tag{1.37}$$

The magnetic moment μ is adiabatically conserved in any magnetic field structure in which the variation of the external fields are slow and on larger scales than the fast gyromotion. Hence $\nabla \mu = 0$ and we get the so-called **gradient-B drift** that follows from this force

Gradient-B drift

$$\mathbf{F}_{\nabla B} = -\mu \nabla B \qquad \Rightarrow \qquad \mathbf{v}_{\nabla B} = -\frac{\mu c}{Ze} \frac{\nabla B \times \mathbf{B}}{B^2}. \tag{1.38}$$

This drift can be understood in the following way: When a particle gyrates in a magnetic field that is inhomogeneous, the gyrofrequency $\omega_c = eZB/(mc)$ will change. With the gyrofrequency, the Larmor radius changes as well. Hence, the particle gyrates in and out of the region of stronger magnetic field strength and drifts in the direction perpendicular to the magnetic field gradient and the magnetic field. It is a similar explanation to the $E \times B$ drift, where parts of the orbit the particle accelerates and in others it decelerates which affects the Larmor radius via the perpendicular velocity.

Note that since the drift velocity depends on the sign (and the force doesn't) the drift direction is different for positively and negatively charged particles.

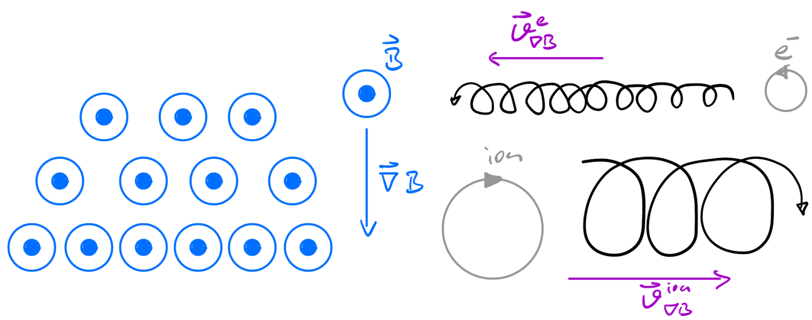

Figure 1.6: Sketch of the gradient-B drift. The particle gyrate in a magnetic field that is inhomogeneous.

This is sketched in figure 1.6.

The gradient-B force is not exclusive to the perpendicular direction. Therefore, we also get a force parallel to the magnetic field. This force is called the **mirror force** and is given by

$$F_{\nabla B,\parallel} = -\mu \nabla_{\parallel} B. \tag{1.39}$$

This force accelerates or decelerates the parallel motion of the particle. But why is it called mirror force? Consider a particle with a magnetic moment in an external magnetic field that is inhomogeneous. Now, we follow the inhomogeneous magnetic field from a region of lower field strength to larger field strength. At a certain point, the particle will be decelerated to zero parallel velocity and reflected such that the sign of the parallel velocity changes. This is called the **mirror effect**.

Instead of the Newtonian force picture, we can consider the Hamiltonian energy picture for a second. With the definition of the magnetic moment, we can rewrite the Hamiltonian as

$$H = \frac{mv^2}{2} + Ze\Phi \tag{1.40}$$

$$= \frac{mv_{\parallel}^2}{2} + \frac{mv_{\perp}^2}{2} + Ze\Phi \tag{1.41}$$

$$= \frac{mv_{\parallel}^2}{2} + \mu B + Ze\Phi. \tag{1.42}$$

Then, we can express the parallel velocity in terms of the conserved energy and magnetic moment as

$$v_{\parallel} = \pm\sqrt{\frac{2}{m}\Big(H - \mu B\Big)}, \tag{1.43}$$

where we neglect the electrostatic potential for a minute. Since energy and magnetic moment are conserved and the velocity cannot become complex, there

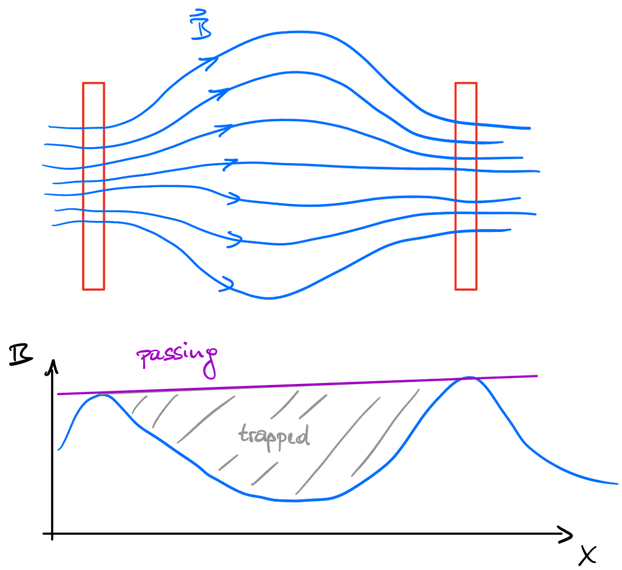

Figure 1.7: Sketch of the magnetic mirror effect. Indicated in red are circular coils that generate the magnetic field given in blue. The magnetic field strength is largest in the center of the coils and decreases towards the middle.

is no possibility that the particle enters regions of sufficiently large magnetic field strength. Otherwise $v_\parallel$ would become complex. Rather, the particle is reflected from this region. The particle is trapped in a region of low magnetic field strength bounded by "walls" of larger magnetic field strength. This is sketched in figure 1.7. Particles with enough energy can escape the magnetic mirror and are called passing particles.

In toroidal magnetic confinement devices, the magnetic mirror effect leads to so-called **banana orbits** because the magnetic field strength is stronger closer to the center of the device. These are orbits that when considering cross-sections of the torus will trace out banana-shaped curves. A sketch of this is shown in figure 1.8.

Another ubiquitous quantity occurring in plasma physics is the **adiabatic invariant**. This quantity is related to the magnetic moment by

$$\omega_c J_\perp = \mu B. \tag{1.44}$$

There is a second drift related to a non-uniform magnetic field. It is called the **curvature drift** and stems from the curvature of magnetic field lines. A particle following a magnetic field line that is curved will feel a centrifugal force

$$\mathbf{F}_c = -mv_\parallel^2 \frac{\mathbf{R}}{R^2}, \tag{1.45}$$

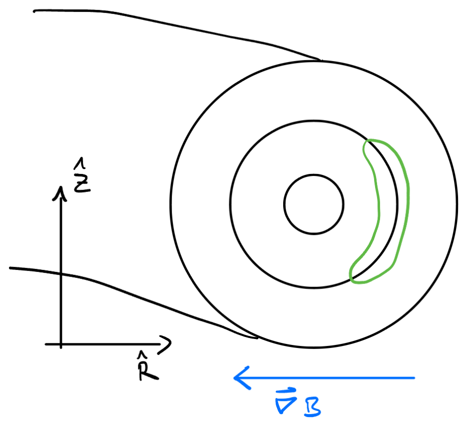

Figure 1.8: Sketch of a banana orbit in circular cross-section tokamak. The magnetic mirror effect occurs because the magnetic field is larger on the inside of the torus.

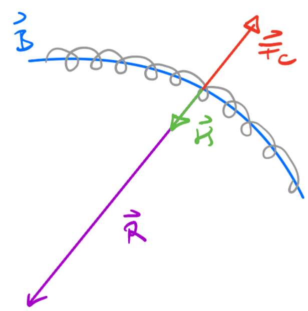

Figure 1.9: Sketch of a particle following a curved magnetic field line.

where $\mathbf{R}$ is the radius vector pointing inward to the center of the magnetic field curve from the particle as sketched in figure 1.9. Here, $\mathbf{R}/R^2$ represents the curvature usually written as $\boldsymbol{\kappa}$. For a curved magnetic field, this vector is given by

$$\boldsymbol{\kappa} = \mathbf{h} \cdot \nabla \mathbf{h}. \tag{1.46}$$

The curvature drift is then given by

Curvature drift

$$\mathbf{v}_\kappa = -\frac{v_\parallel^2}{\omega_c} \frac{\boldsymbol{\kappa} \times \mathbf{B}}{B}. \tag{1.47}$$

This drift again depends on the sign of the charge via ω_c. Also, curvature

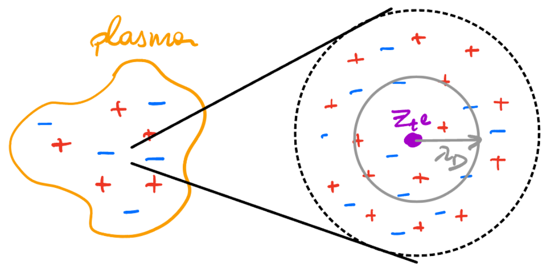

Figure 1.10: Illustration of the quasineutrality with a zoom to the Debye length scale.

drift can only appear if there is a gradient in the magnetic field strength. Hence, curvature drift only occurs if also gradient drifts are present.

Note that there are further drifts, for example the polarization drift that occurs due to a time dependence in the electric field or a finite Larmor radius addition to the $E \times B$ drift due to a non-uniform electric field. However, we will not consider these drifts here. For more information, see any standard plasma physics text book, e.g. [7, 1].

1.3 From particles to plasma

As mentioned in the introduction, a plasma is a partially or fully ionized gas and so it is comprised of many charged particles. Up to this point we discussed the motion of a single charged particle in electromagnetic fields. In addition to the particle behavior of the plasma, it shows collective effects that are absent in neutral gases. Therefore, let's talk now about the plasma as a whole and the collectivity.

1.3.1 Collective behavior

An assumption we usually make about plasma is **quasineutrality**. In other words, we assume the plasma to be always overall neutral. Hence,

$$\sum_{\sigma} Z_{\sigma} e n_{\sigma} = 0, \tag{1.48}$$

where $n_{\sigma} = n_{\sigma}(\mathbf{x}, t)$ is the particle density of species σ giving the number of particles per volume. For an electron-ion plasma for instance, this would be

$$Z_i e n_i - e n_e = 0 \Rightarrow Z_i n_i = n_e. \tag{1.49}$$

Neutrality, because the plasma is overall neutral, but it is only "quasi" because on small scales the condition is not fulfilled.

The spatial scale on which the neutrality condition is not fulfilled anymore is given by the **Debye length**. This length is determined by the electrostatic potential created by a single charge $Z_t e$ that is surrounded by other charges, as illustrated in figure 1.10. Without giving the explicit derivation which can be found elsewhere [7], the spherically symmetric potential is given by

$$\Phi(r) = \frac{Z_t e}{r} \mathrm{e}^{-\frac{r}{\lambda_D}}, \tag{1.50}$$

where Z_σ is the charge number of the single charge surrounded by the other charges, and

$$\lambda_D = \left(\sum_\sigma \frac{4\pi n_\sigma (Z_\sigma e)^2}{T_\sigma} \right)^{-1/2} \tag{1.51}$$

is the Debye length. Here, T_σ is the temperature of the species. The potential (1.50) tells us two things. The first factor gives the typical Coulomb potential of the test charge. The second factor, however, indicates that charges are shielded or screened on typical length scales greater than $\sim \lambda_D$, which is called **Debye shielding**. Thus, the quasineutrality assumption is invalid for length scales shorter than that. In a typical fusion reactor, it is of the order of 10 μm. In astrophysical plasmas, the Debye length can even be in the range of meters to kilometers.

The time scale on which quasineutrality is restored is given by the **plasma frequency** of electrons,

$$\omega_{pe} = \sqrt{\frac{4\pi e^2 n_e}{m_e}}. \tag{1.52}$$

It determines the time scale on which the electrons can balance out a charge imbalance. For typical fusion plasmas, the frequency is in the range of 10^{13} rad/s, which is usually the largest frequency. Accordingly, the time scale of balancing quasineutrality is of the order 10^{-14} s. The plasma frequency can be determined by considereing the restoring force of electrons that are shifted out of quasineutrality from a positive background. Details can be found elsewhere [7, 1].

The Debye length and the plasma frequency are related by the thermal velocity

$$\lambda_{D\sigma} \omega_{p\sigma} = \sqrt{\frac{T_\sigma}{m_\sigma}} = v_{T\sigma}. \tag{1.53}$$

The thermal velocity of electrons in a typical fusion reactor is roughly 1-10% of the speed of light, while for ions it is much smaller (according to the mass ratio). Note, that often the thermal velocity is defined as $v_T = \sqrt{2T/m}$. Always make sure to know which definition is used.

1.3.2 Self-consistent particle description of plasma

Now, let's build a self-consistent model of a plasma in the particle picture.

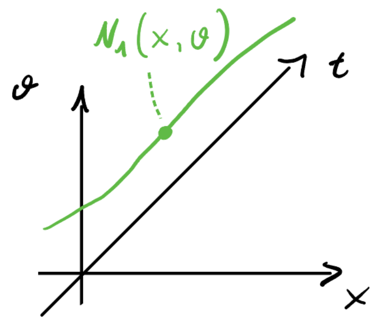

Figure 1.11: Time evolution of a single particle in phase space.

First of all, recall that the particle trajectory $\mathbf{X}(t), \mathbf{V}(t)$ in phase space is governed by the equations of motion

$$\dot{\mathbf{X}}(t) = \mathbf{V}(t), \tag{1.54}$$

$$\dot{\mathbf{V}}(t) = \frac{Z_\sigma e}{m_\sigma}\left(\mathbf{E}(\mathbf{X}(t), t) + \frac{1}{c}\mathbf{V}(t) \times \mathbf{B}(\mathbf{X}(t), t)\right), \tag{1.55}$$

where we use capital letters for a reason that will become apparent in a minute. Of course, the motion of charged particles in electromagnetic fields that was described in the prior section is contained in $\mathbf{X}(t)$ and $\mathbf{V}(t)$. Every drift discussed previously occurs in the Lagrangian coordinates of the particle.

In phase space, the particle is described by the

Single particle phase space density

$$N_{\sigma,1}(\mathbf{x}, \mathbf{v}, t) = \delta(\mathbf{x} - \mathbf{X}(t))\delta(\mathbf{v} - \mathbf{V}(t)), \tag{1.56}$$

where $\delta(\mathbf{x}-\mathbf{X}) = \Pi_{j=1}^{N_D}\delta(x_j - X_j)$ with the number of dimensions N_D. Note that this definition of the particle density is specific for the particle picture. The kinetic (and fluid) pictures use a different definition which we shall see later. The "single particle density" in 2D phase space is sketched in figure 1.11.

A word to the difference in capitalization. The coordinates $\mathbf{x}$ and $\mathbf{v}$ are called the **Eulerian frame** of reference, while $\mathbf{X}$ and $\mathbf{V}$ are called **Lagrangian frame**. The former is a "stationary" frame, i.e. looking at a fixed point of phase space from a fixed point of observation. The latter is a "co-moving" frame, i.e. it is fixed to a point of phase space that evolves with time, which in this case is the trajectory of the charged particle.

So far, we only considered the motion of a charged particle in fixed electromagnetic fields. However, fields are also generated by the particle itself. On the one hand, this increases the complexity of the problem manifold, in particular, if more than one particle is considered. On the other hand, this is what introduces the collective behavior of the plasma and makes it so rich in phenomena.

The electromagnetic fields are determined by Maxwell's equations

Maxwell's equations

$$\nabla \cdot \mathbf{E}(\mathbf{x},t) = 4\pi\rho(\mathbf{x},t) \quad \text{Gauss' law,} \tag{1.57}$$

$$\nabla \cdot \mathbf{B}(\mathbf{x},t) = 0 \quad \text{Source-freeness of } \mathbf{B}, \tag{1.58}$$

$$\nabla \times \mathbf{E}(\mathbf{x},t) = -\frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t} \quad \text{Faraday's law,} \tag{1.59}$$

$$\nabla \times \mathbf{B}(\mathbf{x},t) = \frac{4\pi}{c}\mathbf{j}(\mathbf{x},t) + \frac{1}{c}\frac{\partial \mathbf{E}(\mathbf{x},t)}{\partial t} \quad \text{Ampere's law.} \tag{1.60}$$

Here, ρ and j are the charge and current densities. There are two types of electromagnetic fields: the externally generated ones and the ones provided by the particle itself. The latter are determined by providing expressions for ρ and j given the charged particle.

The charge and current densities are given by velocity moments over the particle density

Charge and current density of a single particle

$$\begin{aligned}\rho(\mathbf{x},t) &= Z_\sigma e \int \mathrm{d}v^3 \; N_{\sigma,1}(\mathbf{x},\mathbf{v},t) \\ &= Z_\sigma e \delta(\mathbf{x} - \mathbf{X}(t))\end{aligned} \tag{1.61}$$

$$\begin{aligned}\mathbf{j}(\mathbf{x},t) &= Z_\sigma e \int \mathrm{d}v^3 \; \mathbf{v} N_{\sigma,1}(\mathbf{x},\mathbf{v},t) \\ &= Z_\sigma e \mathbf{V}(t) \delta(\mathbf{x} - \mathbf{X}(t)).\end{aligned} \tag{1.62}$$

Together with Maxwell's equations, these two definitions give a self-consistent model of a single charged particle in electromagnetic fields.

Generalising to a plasma, i.e. a set of many particles, the main difference lies in the definition of the phase space particle density. The phase space particle density is a sum of the single particle density

$$N_\sigma(\mathbf{x},\mathbf{v},t) = \sum_{i=1}^{N_\sigma^{\mathrm{tot}}} N_{\sigma,1}^i(\mathbf{x},\mathbf{v},t) \tag{1.63}$$

$$= \sum_{i=1}^{N_\sigma^{\mathrm{tot}}} \delta(\mathbf{x} - \mathbf{X}_i(t))\delta(\mathbf{v} - \mathbf{V}_i(t)), \tag{1.64}$$

where N_σ^{tot} is the total particle number of the species and i is the index of the particle. Of course, the total particle density is given by the sum over species

$$N(\mathbf{x}, \mathbf{v}, t) = \sum_{\sigma} N_{\sigma}(\mathbf{x}, \mathbf{v}, t). \tag{1.65}$$

Similar definitions of the charge and current density lead to a self-consistent set of equations for the whole plasma.

However, consider a typical fusion plasma with about 10^{20} charged particles. In practical applications, we are interested in quantities that we can measure, like the bulk flow of a plasma or its density. To determine such macroscopic quantities from the particle picture, we would have to solve for the trajectory $(\mathbf{X}(t),\ \mathbf{V}(t))$ of every single particle! This is impossible. This is why we need a different formulation of plasma that still incorporates the particle nature of plasma but lets us determine macroscopic quantities. Here, kinetic theory comes into play.

1.4 Questions Plasma Physics

Introduction

(1.1) What different descriptions of plasma are there? How are they related conceptually?

(1.2) What is the difficulty of the single particle description?

(1.3) What is the downside of the fluid description?

(1.4) What is the advantage of kinetic theory?

Particle picture

(1.5) What is the Larmor radius and the cyclotron frequency, how are they related, and why are they important?

(1.6) Explain the motion of a charged particle in constant electric and magnetic fields.

(1.7) What particle drifts do you know? Explain. How are they generalized?

(1.8) What is an ambipolar drift?

(1.9) What is the magnetic moment?

(1.10) What quantities are conserved for the particle motion?

(1.11) Describe the magnetic mirror effect. What type of orbits arise from this effect?

The full particle picture of plasma

(1.12) What is the quasineutrality condition? Why is it only ”quasi”?

(1.13) What is Debye shielding?

(1.14) What is the phase space particle density of the particle picture? Explain its components and the difference between the Eulerian frame and the Lagrangian frame.

(1.15) What equations are required to self-consistently describe a single particle in electromagnetic fields?

(1.16) How can this be generalized to more particles?

Chapter 2

Kinetic Theory

As mentioned in the introduction, kinetic theory is a description of the distribution of particles in phase space. In this chapter, we will derive the plasma kinetic equation, discuss its equilibrium solutions, and introduce some basic concepts of kinetic plasma waves.

2.1 Plasma kinetic equation

2.1.1 Derivation via the Klimontovich equation

There are multiple possibilities to derive the plasma kinetic equation (for an extensive derivation consider [7]). Here, we follow a path that seamlessly connects to the previous chapter and is arguably instructive, but is unfortunately not very rigorous mathematically.

Consider a particle species σ. A number of N_σ^{tot} particles are described in phase space $(\mathbf{x}, \mathbf{v})$ by

$$N_\sigma(\mathbf{x}, \mathbf{v}, t) = \sum_{i=1}^{N_\sigma^{\mathrm{tot}}} \delta(\mathbf{x} - \mathbf{X}_i(t))\delta(\mathbf{v} - \mathbf{V}_i(t)), \tag{2.1}$$

where the index indicates the particle that follows its trajectory $\mathbf{X}_i, \mathbf{V}_i$ in 6D phase space. An example of this "spiky" particle density is sketched in figure 2.1

Let's assume that there are no sources of particles, then, the particle density does not change over time,

$$\frac{\mathrm{d}N_\sigma(\mathbf{x}, \mathbf{v}, t)}{\mathrm{d}t} = 0. \tag{2.2}$$

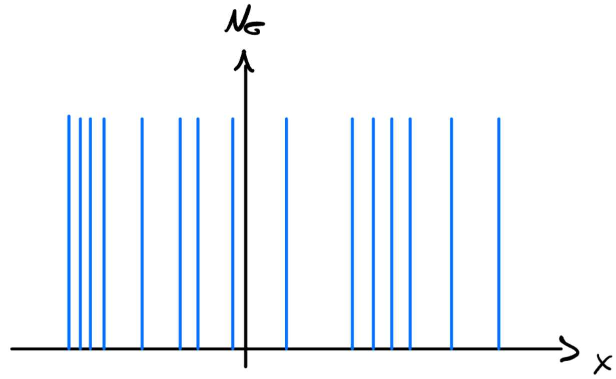

Figure 2.1: Sketch of the particle density function $N_\sigma(\mathbf{x}, \mathbf{v}, t)$ in one spatial dimension.

Now, we start by evaluating the time derivative

$$\frac{\partial N_\sigma(\mathbf{x}, \mathbf{v}, t)}{\partial t} = \sum_i \frac{\partial}{\partial t} \left(\delta(\mathbf{x} - \mathbf{X}_i(t))\delta(\mathbf{v} - \mathbf{V}_i(t))\right) \tag{2.3}$$

$$= \sum_i \delta(\mathbf{v} - \mathbf{V}_i(t)) \frac{\partial}{\partial t}\delta(\mathbf{x} - \mathbf{X}_i(t)) + \sum_i \delta(\mathbf{x} - \mathbf{X}_i(t)) \frac{\partial}{\partial t}\delta(\mathbf{v} - \mathbf{V}_i(t)). \tag{2.4}$$

The time derivative of the Dirac delta can be determined by the chain rule[1]

$$\frac{\partial}{\partial t}\delta(\mathbf{x} - \mathbf{X}_i(t)) = \frac{\partial \delta(\mathbf{x} - \mathbf{X}_i(t))}{\partial(\mathbf{x} - \mathbf{X}_i(t))} \cdot \frac{\partial(\mathbf{x} - \mathbf{X}_i(t))}{\partial t} \tag{2.5}$$

$$= \frac{\partial \delta(\mathbf{x} - \mathbf{X}_i(t))}{\partial x_j} \underbrace{\frac{\partial x_j}{\partial(\mathbf{x} - \mathbf{X}_i(t))_k}}_{\delta_{jk}} \left(-\frac{\partial (X_i)_k}{\partial t}\right) \tag{2.6}$$

$$= \frac{\partial \delta(\mathbf{x} - \mathbf{X}_i(t))}{\partial \mathbf{x}} \cdot \left(-\frac{\partial \mathbf{X}_i}{\partial t}\right) \tag{2.7}$$

$$= -\dot{\mathbf{X}}_i \cdot \nabla_{\mathbf{x}} \delta(\mathbf{x} - \mathbf{X}_i(t)), \tag{2.8}$$

where the Einstein summation convention was used. With this, we can further

[1] The derivative of the Dirac delta function does only make sense for an integral evaluation with a compactly supported smooth test function. However, we assume here, that everything is working fine. In principle, we could multiply the equation with a test function and integrate over it, which would be fine.

write the time derivative of the particle density

$$\frac{\partial N_\sigma(\mathbf{x},\mathbf{v},t)}{\partial t} = \sum_i \delta(\mathbf{v}-\mathbf{V}_i(t))\frac{\partial}{\partial t}\delta(\mathbf{x}-\mathbf{X}_i(t)) + \sum_i \delta(\mathbf{x}-\mathbf{X}_i(t))\frac{\partial}{\partial t}\delta(\mathbf{v}-\mathbf{V}_i(t)) \tag{2.9}$$

$$= -\sum_i \dot{\mathbf{X}}_i \cdot \nabla_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{X}_i(t))\delta(\mathbf{v}-\mathbf{V}_i(t)) - \sum_i \dot{\mathbf{V}}_i \cdot \nabla_{\mathbf{v}}\delta(\mathbf{x}-\mathbf{X}_i(t))\delta(\mathbf{v}-\mathbf{V}_i(t)) \tag{2.10}$$

$$= -\sum_i \mathbf{V}_i \cdot \nabla_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{X}_i(t))\delta(\mathbf{v}-\mathbf{V}_i(t)) - \sum_i \frac{Z_\sigma e}{m_\sigma}\left(\mathbf{E}^m(\mathbf{X}_i(t),t) + \frac{1}{c}\mathbf{V}_i(t)\times\mathbf{B}^m(\mathbf{X}_i(t),t)\right)\cdot\nabla_{\mathbf{v}}\delta(\mathbf{x}-\mathbf{X}_i(t))\delta(\mathbf{v}-\mathbf{V}_i(t)), \tag{2.11}$$

where we have substituted the equations of motion of the single particle for $\dot{\mathbf{X}}$ and $\dot{\mathbf{V}}$ (equations (1.54) and (1.55)). Notice that the electromagnetic fields are evaluated at the position of each individual particle. Also, the electromagnetic fields are the **microscopic fields** that are comprised of the external fields and the fields generated by each particle.

Now, and this is now a less rigorous approach, we can apply the following property of the Dirac delta

$$f(b)\delta(a-b) = f(a)\delta(a-b). \tag{2.12}$$

This property rigorously is only valid when the Dirac delta is under an integral. It is likely possible to multiply the equation with a test function and integrate over it to show that the following result is valid. However, we don't do that here and are content with the handwaving approach. Thus, we have

$$\frac{\partial N_\sigma(\mathbf{x},\mathbf{v},t)}{\partial t} = -\sum_i \mathbf{V}_i \cdot \nabla_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{X}_i(t))\delta(\mathbf{v}-\mathbf{V}_i(t)) - \sum_i \frac{Z_\sigma e}{m_\sigma}\left(\mathbf{E}^m(\mathbf{X}_i(t),t) + \frac{1}{c}\mathbf{V}_i(t)\times\mathbf{B}^m(\mathbf{X}_i(t),t)\right)\cdot\nabla_{\mathbf{v}}\delta(\mathbf{x}-\mathbf{X}_i(t))\delta(\mathbf{v}-\mathbf{V}_i(t)) \tag{2.13}$$

$$= -\sum_i \mathbf{v} \cdot \nabla_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{X}_i(t))\delta(\mathbf{v}-\mathbf{V}_i(t)) - \sum_i \frac{Z_\sigma e}{m_\sigma}\left(\mathbf{E}^m(\mathbf{x},t) + \frac{1}{c}\mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)\right)\cdot\nabla_{\mathbf{v}}\delta(\mathbf{x}-\mathbf{X}_i(t))\delta(\mathbf{v}-\mathbf{V}_i(t)) \tag{2.14}$$

$$= -\mathbf{v} \cdot \nabla_{\mathbf{x}} \sum_i \delta(\mathbf{x} - \mathbf{X}_i(t))\delta(\mathbf{v} - \mathbf{V}_i(t))$$
$$- \frac{Z_\sigma e}{m_\sigma} \left(\mathbf{E}^m(\mathbf{x}, t) + \frac{1}{c}\mathbf{v} \times \mathbf{B}^m(\mathbf{x}, t) \right) \cdot \nabla_{\mathbf{v}} \sum_i \delta(\mathbf{x} - \mathbf{X}_i(t))\delta(\mathbf{v} - \mathbf{V}_i(t)) \tag{2.15}$$

$$= -\mathbf{v} \cdot \nabla_{\mathbf{x}} N_\sigma(\mathbf{x}, \mathbf{v}, t) - \frac{Z_\sigma e}{m_\sigma} \left(\mathbf{E}^m(\mathbf{x}, t) + \frac{1}{c}\mathbf{v} \times \mathbf{B}^m(\mathbf{x}, t) \right) \cdot \nabla_{\mathbf{v}} N_\sigma(\mathbf{x}, \mathbf{v}, t), \tag{2.16}$$

and we finally arrive at

Klimontovich's equation

$$\frac{\partial N_\sigma}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} N_\sigma + \frac{Z_\sigma e}{m_\sigma} \left(\mathbf{E}^m(\mathbf{x}, t) + \frac{1}{c}\mathbf{v} \times \mathbf{B}^m(\mathbf{x}, t) \right) \cdot \nabla_{\mathbf{v}} N_\sigma = 0. \tag{2.17}$$

Let's discuss this equation for a moment. First of all, this equation is still **exact** and provides the evolution of the particle density in 6D phase space. Together with Maxwell's equations, the Klimontovich equation represents an exact description of plasma. That is, with the knowledge of the initial positions and velocities of all the particles, the initial particle density is $N(\mathbf{x}, \mathbf{v}, t = 0)$ is given. From this initial particle density, Maxwell's equations determines the initial microscopic fields (in addition to possible external fields). However, this is not practical since the Klimontovich equation contains the phase space trajectory of every single one of the particles. Apart from being too complex to be solved anyways, we are not interested in this kind of information. Rather, we care for average properties, for example a long-range electric field. In this sense, the Klimontovich is most useful as the starting point for deriving an equation describing average properties of the plasma.

To get to such an "averaged" description, let's consider the particle density N_σ for a moment. It tells us if a particle (with infinite density) can be found at point $(\mathbf{x}, \mathbf{v})$ in phase space. Hence, at every point in phase space where a particle can be found, there is a Dirac-delta spike in N_σ. A more practical approach is to ask the question how many particles are likely to be found in a unit of phase space volume $\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$ centered around $\mathbf{x}, \mathbf{v}$. This requires an **ensemble averaging** of the particle density,

$$f_\sigma(\mathbf{x}, \mathbf{v}, t) = \langle N(\mathbf{x}, \mathbf{v}, t) \rangle. \tag{2.18}$$

That is, an averaging over infinite realizations of the plasma which is prepared according to some description [7].

The quantity $f_\sigma(\mathbf{x}, \mathbf{v}, t)$ is called (phase space or particle) **distribution function**. As its name says, it tells us about how the particles are distributed over phase space. An illustration of the particle density function and the resulting ensemble average is sketched in figure 2.2. The number of particles of

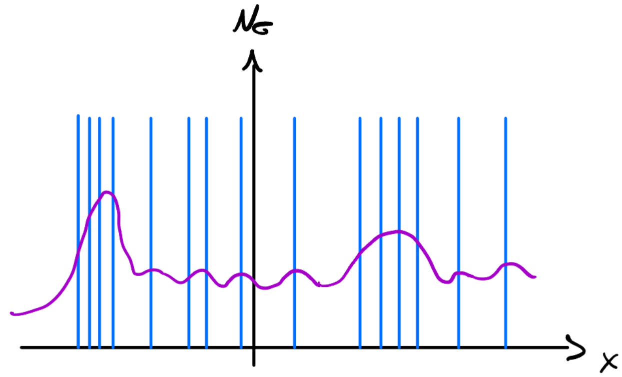

Figure 2.2: Sketch of the particle density function $N_\sigma(\mathbf{x}, \mathbf{v}, t)$ in one spatial dimension and the resulting ensemble average.

species σ at time t in the unit of phase space volume given by $\mathbf{x}$ to $\mathbf{x} + \Delta\mathbf{x}$ and $\mathbf{v}$ to $\mathbf{v} + \Delta\mathbf{v}$ is given by

$$\mathrm{d}N_\sigma = f_\sigma(\mathbf{x}, \mathbf{v}, t)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}. \tag{2.19}$$

From this, the unit of f is $\mathrm{s}^3\mathrm{cm}^{-6}$. It is clear, that f has to be finite, continuous and positive for all values of t. Moreover, in homogeneous plasma, f is independent of $\mathbf{x}$, and for isotropic plasma, f is independent of the direction of $\mathbf{v}$, while in anisotropic plasma f will depend on the orientation of $\mathbf{v}$.

To arrive at an equation that governs the dynamics of the distribution function we ensemble average Klimontovich's equation. However, care has to be taken because the particle density as well as the microscopic electromagnetic fields are not equal to their ensemble averages but they are affected by fluctuations. Therefore, we have to write

$$N_\sigma(\mathbf{x}, \mathbf{v}, t) = f_\sigma(\mathbf{x}, \mathbf{v}, t) + \delta N_\sigma(\mathbf{x}, \mathbf{v}, t), \tag{2.20}$$

$$\mathbf{E}^m(\mathbf{x}, t) = \mathbf{E}(\mathbf{x}, t) + \delta\mathbf{E}(\mathbf{x}, t) \tag{2.21}$$

$$\mathbf{B}^m(\mathbf{x}, t) = \mathbf{B}(\mathbf{x}, t) + \delta\mathbf{B}(\mathbf{x}, t). \tag{2.22}$$

Here, $\mathbf{E} = \langle \mathbf{E}^m \rangle$ and $\mathbf{B} = \langle \mathbf{B}^m \rangle$ are the ensemble-averaged, or macroscopic fields and δN_σ, $\delta\mathbf{E}$ and $\delta\mathbf{B}$ are the fluctuations. It holds that $\langle \delta N_\sigma \rangle = \langle \delta\mathbf{E} \rangle = \langle \delta\mathbf{B} \rangle = 0$.

Let's average Klimontovich's equation,

$$\frac{\partial \langle N_\sigma \rangle}{\partial t} + \mathbf{v} \cdot \nabla \langle N_\sigma \rangle + \frac{Z_\sigma e}{m_\sigma} \left\langle \left(\mathbf{E}^m + \frac{1}{c}\mathbf{v} \times \mathbf{B}^m \right) \cdot \nabla_{\mathbf{v}} N_\sigma \right\rangle = 0. \tag{2.23}$$

Evaluating the Lorentz force term has to be done with care. Since the averaging operation is linear, let's just consider the electric field term first,

$$\langle \mathbf{E}^m \cdot \nabla_{\mathbf{v}} N_\sigma \rangle = \langle (\mathbf{E} + \delta\mathbf{E}) \cdot \nabla_{\mathbf{v}} (f_\sigma + \delta N_\sigma) \rangle \tag{2.24}$$

$$= \langle \mathbf{E} \cdot \nabla_{\mathbf{v}} f_\sigma + \delta\mathbf{E} \cdot \nabla_{\mathbf{v}} f_\sigma + \mathbf{E} \cdot \nabla_{\mathbf{v}} \delta N_\sigma + \delta\mathbf{E} \cdot \nabla_{\mathbf{v}} \delta N_\sigma \rangle. \tag{2.25}$$

Here, the second and third terms vanish, since the ensemble average has no effect on the already averaged quantities and the average over the fluctuation vanishes. Thus,

$$\langle \mathbf{E}^m \cdot \nabla_{\mathbf{v}} N_\sigma \rangle = \langle \mathbf{E} \cdot \nabla_{\mathbf{v}} f_\sigma + \delta\mathbf{E} \cdot \nabla_{\mathbf{v}} \delta N_\sigma \rangle \tag{2.26}$$

$$= \mathbf{E} \cdot \nabla_{\mathbf{v}} f_\sigma + \langle \delta\mathbf{E} \cdot \nabla_{\mathbf{v}} \delta N_\sigma \rangle. \tag{2.27}$$

A similar result can be obtained for the magnetic field term,

$$\langle \mathbf{v} \times \mathbf{B}^m \cdot \nabla_{\mathbf{v}} N_\sigma \rangle = \mathbf{v} \times \mathbf{B} \cdot \nabla_{\mathbf{v}} f_\sigma + \langle \mathbf{v} \times \delta\mathbf{B} \cdot \nabla_{\mathbf{v}} \delta N_\sigma \rangle. \tag{2.28}$$

Finally, we arrive at the sought-after

Plasma kinetic equation

$$\frac{\partial f_\sigma}{\partial t} + \mathbf{v} \cdot \nabla f_\sigma + \frac{Z_\sigma e}{m_\sigma}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) \cdot \nabla_{\mathbf{v}} f_\sigma = -\frac{Z_\sigma e}{m_\sigma}\left\langle (\delta\mathbf{E} + \frac{1}{c}\mathbf{v} \times \delta\mathbf{B}) \cdot \nabla_{\mathbf{v}} \delta N_\sigma \right\rangle. \tag{2.29}$$

This equation is also called the Vlasov-Boltzmann equation. It governs the evolution of the distribution function in 6D phase space. The left hand side of the equation varies only smoothly in phase space and governs the collective behavior of the plasma represented by the macroscopic electromagnetic fields. The right hand side, however, is susceptible to the discrete-particle nature of the plasma and quantifies collisions, which is represented by the fluctuations in the electromagnetic fields. Often, the right hand side is abbreviated as $(\partial f/\partial t)_{\mathrm{coll.}}$.

2.2 Vlasov equation

We will come back to collisions in section 2.5. For now, we neglect the right hand side. In this case, we have the

Vlasov equation

$$\frac{\partial f_\sigma}{\partial t} + \mathbf{v} \cdot \nabla f_\sigma + \frac{Z_\sigma e}{m_\sigma}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{v}} f_\sigma = 0, \tag{2.30}$$

or collisionless Boltzmann equation. This equation is rich in phenomenology and we will spend a great deal investigating it. Neglecting collisions is very well justified if the collision frequency is small compared to typical frequencies of the collective behavior. Since the collision frequency decreases with increasing temperature this is often a reasonable assumption for fusion-relevant plasmas which have high temperatures.

The fields $\mathbf{E}$ and $\mathbf{B}$ appearing in the Vlasov equation are the averaged fields. They must satisfy Maxwell's equations

$$\nabla \cdot \mathbf{E}(\mathbf{x}, t) = 4\pi\rho(\mathbf{x}, t), \tag{2.31}$$

$$\nabla \cdot \mathbf{B}(\mathbf{x}, t) = 0, \tag{2.32}$$

$$\nabla \times \mathbf{E}(\mathbf{x}, t) = -\frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{x}, t)}{\partial t}, \tag{2.33}$$

$$\nabla \times \mathbf{B}(\mathbf{x}, t) = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{2.34}$$

Here, the charge and current densities are the averaged versions of the microscopic ones, i.e. they are given by velocity moments over the distribution function

$$\rho(\mathbf{x}, t) = \langle \rho^m \rangle = \sum_\sigma Z_\sigma e \int \mathrm{d}^3v \; f_\sigma(\mathbf{x}, \mathbf{v}, t), \tag{2.35}$$

$$\mathbf{j}(\mathbf{x}, t) = \langle \mathbf{j}^m \rangle = \sum_\sigma Z_\sigma e \int \mathrm{d}^3v \; \mathbf{v} f_\sigma(\mathbf{x}, \mathbf{v}, t). \tag{2.36}$$

All in all, the Vlasov equation (2.30), Maxwell's equations and the charge and current density definitions represent a closed set of equations that govern the evolution of the plasma.

Interpretation of the Vlasov equation as flow in phase space

The Vlasov equation can be thought of as the flow of an incompressible fluid in phase space. In that regard, the distribution function f_σ can be thought of as a probability density. Since particles are neither created nor destroyed, the fluid must satisfy a continuity equation

$$\frac{\partial f_\sigma(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \nabla_\mathbf{x} \cdot \left(\left.\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|_{\mathrm{orbit}} f_\sigma(\mathbf{x}, \mathbf{v}, t) \right) + \nabla_\mathbf{v} \cdot \left(\left.\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right|_{\mathrm{orbit}} f_\sigma(\mathbf{x}, \mathbf{v}, t) \right) = 0, \tag{2.37}$$

where $\mathrm{d}/\mathrm{d}t|_{\mathrm{orbit}}$ indicates the time derivative with respect to the orbit of the fluid element. To make the resemblance to the continuity equation clear, we introduce the 6D phase space coordinates $\mathbf{z} = (\mathbf{x}, \mathbf{v})$ and write the equation as

$$\frac{\partial f_\sigma(\mathbf{z}, t)}{\partial t} + \nabla_\mathbf{z} \cdot \left(\left.\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t}\right|_{\mathrm{orbit}} f_\sigma(\mathbf{z}, t) \right) = 0. \tag{2.38}$$

However, the "fluid" represents the probability density of particles and, hence, the orbit of the fluid element must be the same as the orbit of a particle. Thus,

$$\left.\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|_{\mathrm{orbit}} = \mathbf{v} \tag{2.39}$$

$$\left.\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right|_{\mathrm{orbit}} = \frac{Z_\sigma e}{m_\sigma}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B} \right), \tag{2.40}$$

and we again arrive at the Vlasov equation (2.30).

From the distribution function to macroscopic quantities
When experiments are conducted, quantities like the particle density or temperature are measured. How is the distribution function related to these quantities? By taking moments over velocity space. For instance, the particle density is defined by

$$n_\sigma(\mathbf{x}, t) = \int \mathrm{d}^3 v \; f_\sigma(\mathbf{x}, \mathbf{v}, t). \tag{2.41}$$

Further quantities will occur down the road. In section 2.4, we derive the macroscopic fluid description from kinetic theory, which is based on building moments of Vlasov's equation.

2.2.1 Equilibrium solution of the Vlasov equation

Often, when working with the Vlasov equation, we need to know the equilibrium solution, that is the solution in a steady-state ($\partial_t f = 0$). The Vlasov equation can determine solutions of the plasma kinetic equation for time scales that is short compared to the collision time. Note to keep in mind that the equilibrium solution is not necessarily stable. In the following, we discuss a general property of the equilibrium solution and examine a few examples.

Consider the Vlasov equation in terms of a total time derivative along a particle orbit determined by the Lorentz force (for now we omit the species index)

$$\begin{aligned}
\frac{Df(\mathbf{x}, \mathbf{v}, t)}{Dt} &= \frac{\partial f}{\partial t} + \frac{\mathrm{d}\mathbf{X}(t)}{\mathrm{d}t} \cdot \nabla_{\mathbf{x}} f + \frac{\mathrm{d}\mathbf{V}(t)}{\mathrm{d}t} \cdot \nabla_{\mathbf{v}} f & (2.42)\\
&= \frac{\partial f}{\partial t} + \left.\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|_{\mathrm{orbit}} \cdot \nabla_{\mathbf{x}} f + \left.\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right|_{\mathrm{orbit}} \cdot \nabla_{\mathbf{v}} f & (2.43)\\
&= \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{Ze}{m} \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \nabla_{\mathbf{v}} f \\
&= 0. & (2.44)
\end{aligned}$$

This derivative along a particle orbit is often called **convective derivative**. If, along the orbit, there are conserved quantities $C_i(\mathbf{x}, \mathbf{v}, t)$, i.e. constants of motion, we can construct solutions to the Vlasov equation as [7]

$$\frac{D}{Dt} f(\{C_i(\mathbf{x}, \mathbf{v}, t)\}) = \sum_i \frac{\partial f}{\partial C_i} \frac{DC_i}{Dt} = 0, \tag{2.45}$$

since for the constants of motions it holds that $DC_i/Dt = 0$. Therefore, any distribution function that depends only on the constants of motion of individual particle orbits solves Vlasov's equations.

Examples

- $\mathbf{E} = \mathbf{B} = 0$: In this case, the particle is not subject to acceleration, i.e. $\dot{\mathbf{v}} = 0$. As conserved quantities we have

$$H = \frac{mv^2}{2} \tag{2.46}$$

$$\mathbf{p} = m\mathbf{v}, \tag{2.47}$$

  i.e. the energy and momentum of the particle. Thus, the solution to Vlasov's equation is

$$f = f(v_x, v_y, v_z). \tag{2.48}$$

  We can easily check this since

$$\mathbf{v} \cdot \nabla f = 0, \tag{2.49}$$

  which is the only term left in this case.

- $\mathbf{E} = 0$, $\mathbf{B} = B\mathbf{e}_z$: In this case, the constants of motion are the perpendicular energy (or magnetic moment) and the momentum in z direction,

$$W_\perp = \frac{mv_\perp^2}{2} = \mu B, \tag{2.50}$$

$$p_z = mv_z. \tag{2.51}$$

  The equilibrium distribution function is thus

$$f = f(v_\perp, v_z). \tag{2.52}$$

  Let's check this.

$$0 = \mathbf{v} \cdot \nabla f(v_\perp, v_z) + \frac{Ze}{mc}\mathbf{v} \times \mathbf{B} \cdot \nabla_{\mathbf{v}} f(v_\perp, v_z) \tag{2.53}$$

$$= \omega_c \mathbf{v}_\perp \cdot \nabla_{\mathbf{v}} f \tag{2.54}$$

$$= \omega_c \left(v_y \frac{\partial v_\perp}{\partial v_x} \frac{\partial f}{\partial v_\perp} - v_x \frac{\partial v_\perp}{\partial v_y} \frac{\partial f}{\partial v_\perp} \right) \tag{2.55}$$

$$= \omega_c \left(\frac{v_y v_x}{v_\perp} \frac{\partial f}{\partial v_\perp} - \frac{v_x v_y}{v_\perp} \frac{\partial f}{\partial v_\perp} \right) \tag{2.56}$$

$$= 0, \tag{2.57}$$

  where $\mathbf{v}_\perp = \mathbf{v} \times \mathbf{h} = (v_y, -v_x, 0)$ and $\mathbf{h} = \mathbf{B}/B$ is the direction of the magnetic field.

- $\mathbf{B} = 0$, $\mathbf{E} = \mathbf{E}(\mathbf{x}) \neq 0$: If we have an electric field, e.g. one in x direction $\mathbf{E} = -\mathbf{e}_x \mathrm{d}\Phi(x)/\mathrm{d}x$, the constants of motion are

$$p_y = mv_y \tag{2.58}$$

$$p_z = mv_z \tag{2.59}$$

$$H_x = \frac{mv_x^2}{2} + Ze\Phi(x), \tag{2.60}$$

and the equilibrium distribution function is

$$f = f(v_x^2 + 2Ze\Phi/m, v_y, v_z). \tag{2.61}$$

Let's check this. The relevant terms of the Vlasov equation are

$$\mathbf{v} \cdot \nabla f + \dot{\mathbf{v}} \cdot \nabla_{\mathbf{v}} f =$$
$$v_x \frac{\partial}{\partial x} f(v_x^2 + 2Ze\Phi(x)/m, v_y, v_z) + \frac{Ze}{m}\mathbf{E} \cdot \nabla_{\mathbf{v}} f(v_x^2 + 2Ze\Phi(x)/m, v_y, v_z) \tag{2.62}$$
$$= v_x \frac{\partial \Phi}{\partial x}\frac{\partial}{\partial \Phi} f(v_x^2 + 2Ze\Phi(x)/m, v_y, v_z)$$
$$- \frac{Ze}{m}\frac{\partial \Phi}{\partial x}\frac{\partial}{\partial v_x} f(v_x^2 + 2Ze\Phi(x)/m, v_y, v_z). \tag{2.63}$$

Now, let us introduce a utility variable

$$q_x = v_x^2 + 2Ze\Phi, \tag{2.64}$$

with the derivatives

$$\frac{\partial q_x}{\partial \Phi} = \frac{2Ze}{m}, \tag{2.65}$$
$$\frac{\partial q_x}{\partial v_x} = 2v_x. \tag{2.66}$$

Then, we can continue

$$v_x \frac{\partial \Phi}{\partial x}\frac{\partial}{\partial \Phi} f - \frac{Ze}{m}\frac{\partial \Phi}{\partial x}\frac{\partial}{\partial v_x} f = v_x \frac{\partial \Phi}{\partial x}\frac{\partial q_x}{\partial \Phi}\frac{\partial}{\partial q_x} f - \frac{Ze}{m}\frac{\partial \Phi}{\partial x}\frac{\partial q_x}{\partial v_x}\frac{\partial}{\partial q_x} f \tag{2.67}$$
$$= v_x \frac{\partial \Phi}{\partial x}\frac{2Ze}{m}\frac{\partial}{\partial q_x} f - \frac{Ze}{m}\frac{\partial \Phi}{\partial x} 2v_x \frac{\partial}{\partial q_x} f \tag{2.68}$$
$$= 0. \tag{2.69}$$

- We could also write a solution based on the adiabatic invariant introduced in section 1.2.2. This is often done in the area of magnetic confinement fusion.

Maxwell-Boltzmann distribution

A prime example of a solution to the plasma kinetic equation that is often used in practice is the Maxwell-Boltzmann distribution. It describes the spread of velocities in thermal equilibrium. The Maxwell distribution, or just Maxwellian, is given by

Maxwell distribution

$$f_M(\mathbf{v}) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m\mathbf{v}^2}{2k_B T}\right). \tag{2.70}$$

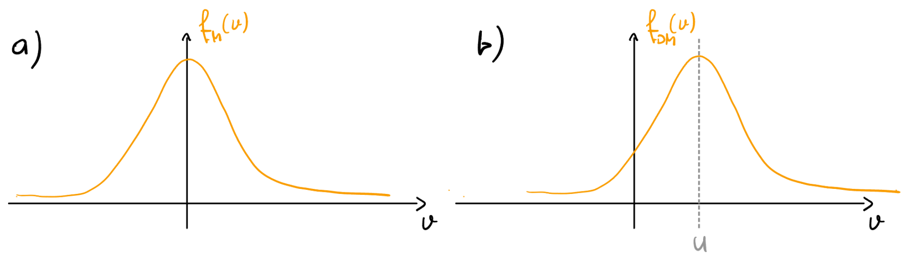

Figure 2.3: Sketch of a) the Maxwell distribution function $f_M(\mathbf{v})$ and b) a drifting Maxwellian in one spatial dimension.

Here, k_B is the Boltzmann constant, n is the macroscopic particle density and T is the temperature. Often, k_B is absorbed into T, which we also do henceforth. Note that the exponent of the bracket varies for different dimensions. Other versions of the distribution function might be drifting ($f \sim \exp(-m(\mathbf{v}-\mathbf{u})^2/(2T))$, or including an electric field ($f \sim \exp(-(m\mathbf{v}^2/2 + Ze\Phi)/T)$). A sketch of the Maxwell distribution is shown in figure 2.3a) and of the drifting Maxwellian in b). The spread of the Gaussian curve of the Maxwellian is defined by the thermal velocity

$$v_T = \sqrt{\frac{T}{m}}, \tag{2.71}$$

that is, the higher the temperature, the larger the spread.

There is also the possibility, that a plasma is only **locally** in thermal equilibrium but not **globally**. This is often the case in fusion plasmas. In this situation, the Maxwellian is local,

$$f_M(\mathbf{x},\mathbf{v}) = n(\mathbf{x})\left(\frac{m}{2\pi k_B T(\mathbf{x})}\right)^{N_D/2} \exp\left(-\frac{m\mathbf{v}^2}{2k_B T(\mathbf{x})}\right). \tag{2.72}$$

It is also often the case, that the different species in a plasma might be in thermal equilibrium within the species, but not with other species. Hence, the temperature is different for the species, i.e. $T_i \neq T_e$.

In the case of spherical symmetry in velocity space (i.e. no preferred direction), we can integrate the angles of the volume element $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$, which gives 4π. In terms of the speed $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$, the distribution function is then given by

$$f(\mathbf{x}, v, t) = 4\pi n(\mathbf{x})\left(\frac{m}{2\pi T(\mathbf{x})}\right)^{N_D} v^2 \exp\left(-\frac{mv^2}{2T(\mathbf{x})}\right). \tag{2.73}$$

In fluid theory, it is assumed that the velocity distribution of the plasma is Maxwellian. This means, that fluid theory neglects effects that depend on variation in the velocity space distribution. For example, including such effects results in the process of Landau damping which is a particle-wave interaction that damps the electro(magnetic) wave by accelerating and decelerating the particles. This will be discussed in more detail in section 2.3.4

2.3 Linear kinetic plasma waves

Linear plasma waves result from a response to small perturbations. The study of plasma waves is an important topic and interesting in itself. It shows how ions, electrons and electromagnetic fields respond differently to a perturbation with the same frequency. In fusion plasmas, they can be used for plasma heating, current drive, diagnostics, or stability control. However, apart from this useful controlled applications, unwanted effects of plasma waves can lead to instabilities. Hence, since plasma waves are truly ubiquitous, we are well advised to study them.

The fluid description of plasma also describes waves. However, since it averages out the velocity space, it neglects certain effects. For example, the resonant wave-particle interaction that results in Landau damping (treated in section 2.3.4), the damping of an electrostatic wave by accelerating particles, does not occur in the fluid description. Nevertheless, fluid theory is well applicable for waves with long wavelengths and low frequency where collisions and collective effects dominate. Consequently, kinetic theory is essential in short-wavelength waves where individual particle dynamics are important.

We will not head on dive into the rabbit hole of plasma waves. Rather, we will dip a toe into a few selected topics that demonstrate important concepts which serves as a base camp from which further exploration into the topic can be started. Most notably, we start with the simplest type of waves, cold plasma waves, in section 2.3.1. This section provides an introduction into the concepts of the dielectric tensor, the conductivity tensor, dispersion relation and so on. Also, it will provide insights into the most prominent plasma waves like Langmuir waves, electron cyclotron wave, whistler wave and more. Then, in section 2.3.2 we loose the assumption of a cold plasma and investigate electrostatic waves and determine the next order correction to Langmuir waves in section 2.3.3. Furthermore, in section 2.3.4 we encounter the prime example of the impact of kinetic theory on plasma waves: Landau damping.

2.3.1 Cold plasma waves

Assumptions

We begin by specifying some basic assumptions. First, we assume a homogeneous steady-state plasma, that is, the distribution function does not depend on space, $f = f(\mathbf{v})$. Further, we **linearize** the distribution function and the electromagnetic fields. Hence, we perturb a homogeneous background as

$$f(\mathbf{x},\mathbf{v},t) = f_0(\mathbf{v}) + \delta f(\mathbf{x},\mathbf{v},t) \tag{2.74}$$
$$\mathbf{E}(\mathbf{x},t) = \delta\mathbf{E}(\mathbf{x},t) \tag{2.75}$$
$$\mathbf{B}(\mathbf{x},t) = \mathbf{B}_0 + \delta\mathbf{B}(\mathbf{x},t). \tag{2.76}$$

Note that we ignore an equilibrium term in the electric field since an electric field can only be present in the perpendicular direction and we are always free to go to a moving reference frame with velocity $\mathbf{v}_E = B_0^{-1}\mathbf{E}_0 \times \mathbf{B}_0$ in which $\mathbf{E}_0 = 0$. Further, we assume that the perturbation is small, implying that

$$f_0 \gg \delta f, \qquad |\mathbf{E}_0| \gg |\delta\mathbf{E}|, \qquad |\mathbf{B}_0| \gg |\delta\mathbf{B}|. \tag{2.77}$$

Also, since we are only interested in linear perturbations we ignore all terms that are of higher order in the perturbation, e.g. δf^2, $\delta f \delta\mathbf{E}$, etc. These would lead to non-linear effects.

A wave solution is retrieved by casting the perturbations in the form

$$\delta f(\mathbf{x},\mathbf{v},t) = \tilde{f}_{\mathbf{k}}(\mathbf{v})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}-\mathrm{i}\omega t} \tag{2.78}$$
$$\delta\mathbf{E}(\mathbf{x},t) = \tilde{\mathbf{E}}_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}-\mathrm{i}\omega t} \tag{2.79}$$
$$\delta\mathbf{B}(\mathbf{x},t) = \tilde{\mathbf{B}}_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}-\mathrm{i}\omega t}. \tag{2.80}$$

In the following, we suppress the index $\mathbf{k}$ of the wave amplitude for brevety. But, keep in mind that the amplitude is specific for each wave vector $\mathbf{k}$.

Cold means that the particles do not have kinetic thermal motion on their own, that is, the phase velocity of the plasma is much larger than the typical speed of a particle [8],

Cold plasma assumption

$$v_p = \frac{\omega}{k} \gg v_{T\sigma} \tag{2.81}$$

for all species σ. This condition implies that all the particles experience the same EM fields. In the other case, that the phase velocity is comparable to the thermal velocity, some particles can leave the wave behind and will see different fields. More about that in section 2.3.4.

When considering waves, what is it that we actually want to determine? A central quantity specifying properties of waves is the **dispersion relation**. It relates the frequency of the wave with the wavelength and tells us how waves with different frequencies travel with different speeds. Also, different kinds of plasma waves, that have different effects like heating or destabilizing, are distinguished by their dispersion relation.

Some further basics about waves can be found in appendix B.

General discussion about the dispersion relation

Using the monochromatic plane wave ansatz, we can write Faraday's law

$$\nabla \times \delta \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\delta \mathbf{B} \tag{2.82}$$

$$\mathbf{k} \times \tilde{\mathbf{E}} = \frac{\mathrm{i}\omega}{c}\tilde{\mathbf{B}} \Rightarrow \tilde{\mathbf{B}} = \frac{c}{\omega}\mathbf{k} \times \tilde{\mathbf{E}}, \tag{2.83}$$

and Ampere's law

$$\nabla \times \delta \mathbf{B} = \frac{4\pi}{c}\delta \mathbf{J} + \frac{1}{c}\frac{\partial}{\partial t}\delta \mathbf{E} \tag{2.84}$$

$$\mathbf{k} \times \tilde{\mathbf{B}} = -\frac{\mathrm{i}4\pi}{c}\tilde{\mathbf{J}} - \frac{\omega}{c}\tilde{\mathbf{E}}, \tag{2.85}$$

as algebraic equations. Inserting the magnetic field from Faraday's law into Ampere's law, we have

$$\mathbf{k} \times (\mathbf{k} \times \tilde{\mathbf{E}}) = -\frac{\mathrm{i}4\pi\omega}{c^2}\tilde{\mathbf{J}} - \frac{\omega^2}{c^2}\tilde{\mathbf{E}} \tag{2.86}$$

which is a wave equation for $\tilde{\mathbf{E}}$. Note that there is an inhomogeneous part to the usual wave equation of the vacuum given by $\tilde{\mathbf{J}}$. This term represents the medium (the plasma) in which the wave propagates. Without that term, we would have a wave with phase velocity $v_p = c$. However, due to this inhomogeneity, we have a wave for which the phase velocity is reduced,

$$v_p = \frac{c}{n}, \tag{2.87}$$

where n is the **index of refraction**. From electrodynamics (see e.g. [3]), we recall that

$$n^2 = \frac{\varepsilon\mu}{\varepsilon_0\mu_0} = \varepsilon_r\mu_r \approx \varepsilon_r \tag{2.88}$$

where ε is the permittivity, μ the permeability and ε_r is the dielectric "constant". In plasma, the relative permeability stays close to 1. Further $\varepsilon_r > 1$, implying that light in matter moves slower than with the speed of light. In general, the dielectric coefficient is not constant, this is in particular true for magnetized plasma in which it is a tensor.

Let's find the dielectric tensor in a general way. We start by relating the current density perturbation and the electric field perturbation as

$$\tilde{\mathbf{J}} = \boldsymbol{\sigma} \cdot \tilde{\mathbf{E}}, \tag{2.89}$$

where $\boldsymbol{\sigma}$ is the **conductivity tensor**. This relation is Ohm's law and this type of relation is also called constitutive relation since it relates a material quantity with a field. Plugging this back into equation (2.86), we get

$$\frac{c^2}{\omega^2}\mathbf{k} \times (\mathbf{k} \times \tilde{\mathbf{E}}) = -\tilde{\mathbf{E}} - \frac{\mathrm{i}4\pi}{\omega}\boldsymbol{\sigma} \cdot \tilde{\mathbf{E}} \tag{2.90}$$

$$= -\boldsymbol{\varepsilon} \cdot \tilde{\mathbf{E}}. \tag{2.91}$$

Here, we introduced the **dielectric tensor**

$$\boldsymbol{\varepsilon} = \mathbb{1} + \frac{\mathrm{i}4\pi}{\omega}\boldsymbol{\sigma}, \tag{2.92}$$

where $\mathbb{1}$ is the unit tensor. How is the dielectric tensor related to the refraction index? Let's get back to the wave equation

$$0 = \frac{c^2}{\omega^2}\mathbf{k} \times (\mathbf{k} \times \tilde{\mathbf{E}}) + \boldsymbol{\varepsilon} \cdot \tilde{\mathbf{E}} \tag{2.93}$$

$$\overset{\text{BAC}-\text{CAB}}{=} \left[\frac{c^2}{\omega^2}(\mathbf{k}\mathbf{k} - k^2\mathbb{1}) + \boldsymbol{\varepsilon}\right] \cdot \tilde{\mathbf{E}} \tag{2.94}$$

$$= \left[\frac{c^2k^2}{\omega^2}(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbb{1}) + \boldsymbol{\varepsilon}\right] \cdot \tilde{\mathbf{E}}, \tag{2.95}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$ is the direction of the wave vector. This equation has the trivial solution $\tilde{\mathbf{E}} = 0$, which is of course not interesting. The interesting solution is found if the other part of the equation is singular,

Dispersion equation

$$\det\left[\frac{c^2k^2}{\omega^2}(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbb{1}) + \boldsymbol{\varepsilon}\right] = 0. \tag{2.96}$$

Solving this equation gives the dispersion relation, i.e. ω as a function of k, depending on the wave direction $\hat{\mathbf{k}}$ and the plasma properties contained in $\boldsymbol{\varepsilon}$. If a plasma is subjected to a perturbation from an antenna with a certain frequency and direction, the plasma responds and will decide k which also allows concluding on the phase velocity [8]. Recall, that the refraction index relates the phase velocity with the speed of light,

$$n = \frac{c}{v_p} = \frac{ck}{\omega}. \tag{2.97}$$

Thus,

$$\det\left[n^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbb{1}) + \boldsymbol{\varepsilon}\right] = 0, \tag{2.98}$$

can also be thought as an equation to determine the index of refraction.

Once the dispersion equation is solved, equation (2.95) gives information about the direction of $\tilde{\mathbf{E}}$. The magnitude of $\tilde{\mathbf{E}}$, however, is not determined by this equation. It is rather defined by the boundary conditions at the antenna.

Dispersion relation excursion: The dispersion relation describes the relation between the frequency and wavenumber of a wave, i.e. $\omega(k)$. In dispersive media, like a plasma, waves with different frequencies will propagate with different phase velocities. As described above, the phase velocity, given by $v_p = \omega(k)/k$ is determined by the dispersion relation. Further, the dispersion

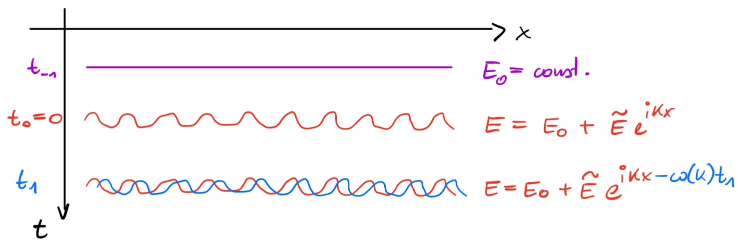

Figure 2.4: Sketch of a linear wave.

relation determines the group velocity, $v_g = \partial\omega/\partial k$. The group velocity is also often thought of the velocity at which the energy of the wave is transported. However, this is not always accurate.

Further, the dispersion relation infers on which types of waves can exist. It helps **distinguish different modes** (waves) and their respective frequency ranges. In that respect, different types of waves only occur for certain frequency ranges. Also, the imaginary part of the dispersion relation, i.e. $\Im(\omega(k))$, indicates the **stability** of the wave. A wave can either be unstable (exponential growth), stable (decay or damping) or it remains constant.

As a trivial example, consider the disperion relation of light waves in vacuum

$$\omega = ck. \tag{2.99}$$

In this case, we have

$$v_p = \frac{\omega}{k} = c, \tag{2.100}$$

$$v_g = \frac{\partial\omega(k)}{\partial k} = c. \tag{2.101}$$

Hence, for light waves in vacuum, the phase and group velocity are equal.

But what does the dispersion relation tell us? Consider the plane wave ansatz for an electric field in one dimension

$$E(x,t) = E_0 + \tilde{E}\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}, \tag{2.102}$$

where E_0 is a possible background field that existed prior to the perturbation. For simplicity, assume it constant. Let's further assume, that we turn the perturbation on at $t_0 = 0$. Then, the electric field is

$$E(x,0) = E_0 + \tilde{E}\mathrm{e}^{\mathrm{i}kx}. \tag{2.103}$$

This is sketched in figure 2.4. At this point, the electric field is given by a constant term and a small modulation in space. Now, going further to a time point t_1 the wave is spatially shifted by $\omega(k)t_1$. The velocity at which the peaks and troughs are moving is the phase velocity $v_p = \omega(k)/k$. Here, we assumed a monochromatic wave that is determined by a single wavenumber k. In general,

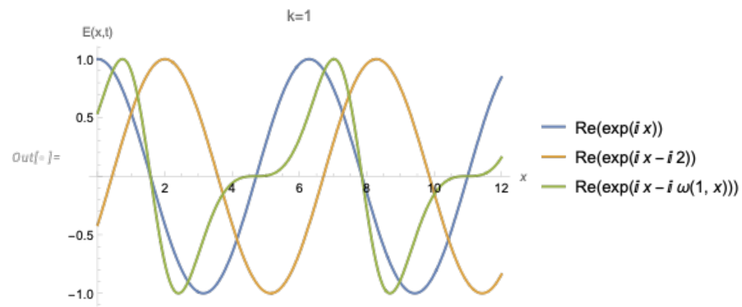

Figure 2.5: Plot of a plane wave in one dimension where the dispersion relation is local, i.e. $\omega = \omega(k, x)$.

a wave can be composed of multiple wavenumbers, i.e. a wave packet. The wave packet is a superposition of monochromatic waves with different wavenumbers. The group velocity is then the velocity at which the wave packet moves.

If the dispersive medium in which the wave oscillates is **inhomogeneous**, the dispersion relation will in general be **local**, i.e. $\omega = \omega(k, x)$. That is, depending on the local parameters like density or temperature in a plasma, the wave will propagate with different phase velocities. This is sketched in figure 2.5. The blue line indicates the wave at $t_0 = 0$. The orange line shows the wave for a later time step with a dispersion relation independent on space. Finally, the green line is the case that $\omega(k) = k\cos(x)$, i.e. it depends on space.

There are **four different types of dispersion relations**:

- $\omega \neq \omega(k)$, dispersive[2]: Oscillating but not propagating. The group velocity is zero. (The wave packet does not move.)
- $\omega(k) \propto k$, non-dispersive: Group and phase velocity are equal. The individual waves of a packet move with the whole packet at the same speed.
- $\omega(k) = ak + b$, dispersive, linear dependence: Group and phase velocity are different. The individual waves of a packet move with different speeds.
- $\omega(k)$, dispersive, non-linear: Group and phase velocity are different and depend on k. The individual waves of a packet move with different speeds and the packet shape changes and gets distorted.

Dielectric tensor derived from Vlasov's equation

In the following, we will derive the dielectric tensor for cold plasma waves from the Vlasov equation and discuss the type of waves that follow. To get the

[2]Dispersive refers to the dependency of the phase velocity on the wavenumber. If it depends on the wave number, it is dispersive.

dielectric tensor we determine the conductivity tensor. For this, we have to get the current density perturbation,

$$\delta\mathbf{j} = \sum_\sigma Z_\sigma e \delta\mathbf{\Gamma}_\sigma, \tag{2.104}$$

which is defined via the flow

$$\delta\mathbf{\Gamma}_\sigma = \int \mathrm{d}^3v\ \mathbf{v}\delta f_\sigma. \tag{2.105}$$

Hence, to determine the perturbation of the flow, we need to consider the perturbation of the distribution function. The linear perturbation of the particle distribution function δf is governed by the linearized Vlasov equation, i.e. inserting the linear expansion of the distribution function and fields in the Vlasov equation

$$\begin{aligned} 0 &= \frac{\partial f_\sigma}{\partial t} + \mathbf{v}\cdot\nabla f_\sigma + \frac{Z_\sigma e}{m_\sigma}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}} f_\sigma && (2.106) \\ &= \cancel{\frac{\partial f_{\sigma,0}}{\partial t}} + \frac{\partial \delta f_\sigma}{\partial t} + \mathbf{v}\cdot\cancel{\nabla f_{\sigma,0}} + \mathbf{v}\cdot\nabla\delta f_\sigma \\ &+ \frac{Z_\sigma e}{m_\sigma}\left(\delta\mathbf{E} + \frac{1}{c}\mathbf{v}\times(\mathbf{B}_0 + \delta\mathbf{B})\right)\cdot\nabla_{\mathbf{v}}(f_{\sigma,0} + \delta f_\sigma). && (2.107) \end{aligned}$$

As mentioned before, we neglect terms of second order in the perturbation. Further, since the equilibrium distribution function is constant in time and homogeneous, the respective terms vanish. Hence, we have

$$\frac{\partial \delta f_\sigma}{\partial t} + \mathbf{v}\cdot\nabla\delta f_\sigma + \frac{Z_\sigma e}{m_\sigma c}(\mathbf{v}\times\mathbf{B}_0)\cdot\nabla_v\delta f_\sigma = -\frac{Z_\sigma e}{m_\sigma}(\delta\mathbf{E} + \frac{1}{c}\mathbf{v}\times\delta\mathbf{B})\cdot\nabla_v f_{\sigma,0}. \tag{2.108}$$

We can further neglect some terms by comparing their relative magnitude

$$\partial_t\delta f \sim \omega\delta f, \tag{2.109}$$

$$\mathbf{v}\cdot\nabla\delta f \sim v_T k\delta f, \tag{2.110}$$

$$\mathbf{v}\times\delta\mathbf{B} \sim v_T\delta B \sim v_t\frac{k}{\omega}\delta E. \tag{2.111}$$

Since we assumed a cold plasma, $v_t \ll \omega/k$, the terms (2.110) and (2.111) are small compared to the first one [3]. Hence, we neglect them and arrive at

$$\frac{\partial \delta f_\sigma}{\partial t} + \frac{Z_\sigma e}{m_\sigma c}(\mathbf{v}\times\mathbf{B}_0)\cdot\nabla_v\delta f_\sigma = -\frac{Z_\sigma e}{m_\sigma}\delta\mathbf{E}\cdot\nabla_v f_{\sigma,0}. \tag{2.112}$$

[3]The magnetic field perturbation magnitude can be estimated by the electric field perturbation magnitude using Faraday's law.

We proceed by multiplying the equation with $\mathbf{v}$ and integrating over velocity space. By partial integration of the two terms on the right, we arrive at

$$\frac{\partial \delta \mathbf{\Gamma}_\sigma}{\partial t} - \omega_{c\sigma} \delta \mathbf{\Gamma}_\sigma \times \mathbf{h} = \frac{Z_\sigma e}{m_\sigma} n_\sigma \delta \mathbf{E}. \tag{2.113}$$

Here, $\omega_{c\sigma} = Z_\sigma e B_0/(m_\sigma c)$ is the cyclotron frequency, $\mathbf{h} = \mathbf{B}_0/B_0$ is the magnetic field direction and $n_\sigma = \int \mathrm{d}^3 v f_0$ is the particle density. This is an equation determining the time evolution of the particle flow, and essentially the current density. Hence, we need to solve it to eventually get an expression for the conductivity tensor provided by Ohm's law.

Since we are interested in waves, we insert the monochromatic wave ansatz and get

$$-\mathrm{i}\omega \tilde{\mathbf{\Gamma}}_\sigma - \omega_{c\sigma} \tilde{\mathbf{\Gamma}}_\sigma \times \mathbf{h} = \frac{Z_\sigma e}{m_\sigma} n_\sigma \tilde{\mathbf{E}}. \tag{2.114}$$

To solve this equation for $\tilde{\mathbf{\Gamma}}_\sigma$ we introduce a coordinate system that is aligned to the magnetic field,

$$\mathbf{e}_1 = \frac{\tilde{\mathbf{E}}_\perp}{\tilde{E}_\perp}, \qquad \mathbf{e}_2 = \mathbf{h} \times \mathbf{e}_1, \qquad \mathbf{e}_3 = \mathbf{h}, \tag{2.115}$$

where $\widetilde{E}_\perp$ is the electric field perpendicular to the magnetic field. By multiplying equation (2.114) with each basis vector, we can write the equation as a matrix vector equation. Multiplying with the first basis vector gives

$$-\mathrm{i}\omega \tilde{\mathbf{\Gamma}} \cdot \mathbf{e}_1 - \omega_{c\sigma} (\tilde{\mathbf{\Gamma}} \times \mathbf{h}) \cdot \mathbf{e}_1 = \frac{Z_\sigma e}{m_\sigma} n_\sigma \tilde{\mathbf{E}} \cdot \mathbf{e}_1 \tag{2.116}$$

$$-\mathrm{i}\omega \tilde{\mathbf{\Gamma}} \cdot \mathbf{e}_1 + \omega_{c\sigma} \tilde{\mathbf{\Gamma}} \cdot (\mathbf{e}_1 \times \mathbf{h}) = \frac{Z_\sigma e}{m_\sigma} n_\sigma \tilde{E}_\perp \tag{2.117}$$

$$-\mathrm{i}\omega \tilde{\mathbf{\Gamma}} \cdot \mathbf{e}_1 - \omega_{c\sigma} \tilde{\mathbf{\Gamma}} \cdot \mathbf{e}_2 = \frac{Z_\sigma e}{m_\sigma} n_\sigma \tilde{E}_\perp. \tag{2.118}$$

This can be repeated with the other two basis vectors. Then, we have the matrix vector equation

$$\begin{pmatrix} -\mathrm{i}\omega & -\omega_{c\sigma} & 0 \\ \omega_{c\sigma} & -\mathrm{i}\omega & 0 \\ 0 & 0 & -\mathrm{i}\omega \end{pmatrix} \cdot \begin{pmatrix} \tilde{\mathbf{\Gamma}}_\sigma \cdot \mathbf{e}_1 \\ \tilde{\mathbf{\Gamma}}_\sigma \cdot \mathbf{e}_2 \\ \tilde{\mathbf{\Gamma}}_\sigma \cdot \mathbf{e}_3 \end{pmatrix} = \frac{Z_\sigma e}{m_\sigma} n_\sigma \begin{pmatrix} \tilde{E}_\perp \\ 0 \\ \tilde{E}_\parallel \end{pmatrix}. \tag{2.119}$$

This equation is solved by matrix inversion (e.g. gaussian elimination, Cayley-Hamilton, etc.). The solution is

$$\begin{pmatrix} \tilde{\mathbf{\Gamma}}_\sigma \cdot \mathbf{e}_1 \\ \tilde{\mathbf{\Gamma}}_\sigma \cdot \mathbf{e}_2 \\ \tilde{\mathbf{\Gamma}}_\sigma \cdot \mathbf{e}_3 \end{pmatrix} = \frac{Z_\sigma e n_\sigma}{m_\sigma} \frac{\mathrm{i}}{\omega(\omega^2 - \omega_{c\sigma}^2)} \begin{pmatrix} \omega^2 \tilde{E}_\perp \\ -\mathrm{i}\omega\omega_{c\sigma} \tilde{E}_\perp \\ (\omega^2 - \omega_{c\sigma}^2) \tilde{E}_\parallel \end{pmatrix}. \tag{2.120}$$

Or in vector notation

$$\tilde{\mathbf{\Gamma}}_\sigma = \frac{Z_\sigma e n_\sigma}{m_\sigma} \mathrm{i}\omega \left(\frac{1}{\omega^2 - \omega_{c\sigma}^2} \tilde{\mathbf{E}}_\perp + \frac{1}{\omega^2} (\tilde{\mathbf{E}} \cdot \mathbf{h}) \mathbf{h} - \frac{\mathrm{i}\omega_{c\sigma}}{\omega(\omega^2 - \omega_{c\sigma}^2)} \mathbf{h} \times \tilde{\mathbf{E}} \right). \tag{2.121}$$

Now we have got the particle flow as a function of the electric field perturbation. In fact, this is a linear relation and we can write the current density as

$$\tilde{\mathbf{J}} = \sum_\sigma Z_\sigma e \tilde{\mathbf{\Gamma}}_\sigma = \boldsymbol{\sigma} \cdot \tilde{\mathbf{E}}. \tag{2.122}$$

We see that we can simply "read off" the conductivity tensor since we know the particle flow. To do so, we realize, that we can write the different vectors occurring in the particle flow as matrix vector products. In particular

$$\tilde{\mathbf{E}}_\perp = \tilde{\mathbf{E}} - \tilde{\mathbf{E}}_\parallel \tag{2.123}$$

$$= \tilde{\mathbf{E}} - \mathbf{h}(\mathbf{h} \cdot \tilde{\mathbf{E}}) \tag{2.124}$$

$$= \underbrace{(\mathbb{1} - \mathbf{hh})}_{\text{matrix}} \cdot \underbrace{\tilde{\mathbf{E}}}_{\text{vector}} . \tag{2.125}$$

Thus, the conductivity operator is

$$\boldsymbol{\sigma} = \mathrm{i}\frac{\omega}{4\pi} \sum_\sigma \left(\frac{\omega_{p\sigma}^2}{\omega^2 - \omega_{c\sigma}^2}(\mathbb{1} - \mathbf{hh}) + \frac{\omega_{p\sigma}^2}{\omega^2}\mathbf{hh} - \frac{\mathrm{i}\omega_{p\sigma}^2 \omega_{c\sigma}}{\omega(\omega^2 - \omega_{c\sigma}^2)} \mathbf{h} \times \mathbb{1} \right), \tag{2.126}$$

where $\omega_{p\sigma} = \sqrt{Z_\sigma^2 e^2 n_\sigma 4\pi / m_\sigma}$ is the plasma frequency. Note that the last term is to be understood as

$$(\mathbf{h} \times \mathbb{1})_{ij} = \varepsilon_{ikl} h_k \delta_{lj} = \varepsilon_{ikj} h_k. \tag{2.127}$$

With this expression for the conductivity tensor, we have determined an expression for the dielectric tensor for cold plasma,

$$\boldsymbol{\varepsilon} = \mathbb{1} + \frac{\mathrm{i}4\pi}{\omega}\boldsymbol{\sigma}. \tag{2.128}$$

The dielectric tensor is the quantity that contains most of the information. Essentially, by encapsulating all particles of the plasma, it tells us how the plasma responds to electric fields depending on the direction, wavenumber and frequency of the wave. It can tell us if the wave propagates in the plasma, grows or if it is damped.

Given the expression for the conductivity tensor, it is useful to split the dielectric tensor into three parts

Dielectric tensor for cold plasma waves

$$\boldsymbol{\varepsilon} = \varepsilon_\perp(\mathbb{1} - \mathbf{hh}) + \varepsilon_\parallel \mathbf{hh} - \mathrm{i}g\mathbf{h} \times \mathbb{1}, \tag{2.129}$$

with the components

$$\varepsilon_\perp = 1 - \sum_\sigma \frac{\omega_{p\sigma}^2}{\omega^2 - \omega_{c\sigma}^2}, \tag{2.130}$$

$$\varepsilon_\parallel = 1 - \sum_\sigma \frac{\omega_{p\sigma}^2}{\omega^2}, \tag{2.131}$$

$$g = -\sum_\sigma \frac{\omega_{p\sigma}^2 \omega_{c\sigma}}{\omega(\omega^2 - \omega_{c\sigma}^2)}. \tag{2.132}$$

It is often useful to distinguish between longitudinal and transverse waves with respect to the background magnetic field direction. For this purpose, instead of aligning the coordinate system with the electric field and the magnetic field, we want to swap the electric field for the wave number, i.e. we have coordinates

$$\hat{\mathbf{x}} = \frac{\mathbf{k}_\perp}{k_\perp}, \qquad \hat{\mathbf{y}} = \mathbf{h} \times \hat{\mathbf{x}}, \qquad \mathbf{h} = \frac{\mathbf{B}_0}{B_0}, \tag{2.133}$$

where $\mathbf{k}_\perp = \mathbf{k} - (\mathbf{k} \cdot \mathbf{h})\mathbf{h}$. In this basis, the dielectric tensor has a nice form [4]

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_\perp & \mathrm{i}g & 0 \\ -\mathrm{i}g & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_\parallel \end{pmatrix}. \tag{2.134}$$

The dielectric tensor is clearly **anisotropic** since $\varepsilon_\perp \neq \varepsilon_\parallel$ and it has **gyrotropic** entries $\propto g$. The gyrotropicity originates from the magnetic field. If an electric field propagates through a gyrotropic medium, the field rotates and thus, the polarization changes. For example, earth's ionosphere is such a gyrotropic medium. As such, it affects the propagation of waves which has to be taken into account in radio and microwave communications with satellites. A linearly polarized wave traveling through a gyrotropic medium will experience a rotation of the linear polarization orientation. This effect is called Faraday rotation. Note that the dielectric tensor is Hermitian [8]. Without proving it here, this makes n^2 real.

Recall, that we want to solve the dispersion equation

$$\det\left(n^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbb{1}) + \boldsymbol{\varepsilon}\right) = 0, \tag{2.135}$$

which gives us the dispersion relation $\omega(k)$.

What can we do now that we have determined the dielectric tensor? We can explore the zoo of plasma waves!

[4]To show this, multiply(2.129) from left and right with the new basis vectors, which gives the elements of the matrix.

Parallel propagation

Let's start with waves that propagate along the magnetic field, i.e. $\hat{\mathbf{k}} = \mathbf{h}$. In this case, the governing equation for the dispersion relation is

$$\left[n^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbb{1}) + \boldsymbol{\varepsilon}\right] \cdot \tilde{\mathbf{E}} = 0 \tag{2.136}$$

$$\left[n^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbb{1}) + \varepsilon_\perp(\mathbb{1} - \mathbf{hh}) + \varepsilon_\parallel \mathbf{hh} - \mathrm{i}g\mathbf{h} \times \mathbb{1}\right] \cdot \tilde{\mathbf{E}} = 0 \tag{2.137}$$

$$\left[(\varepsilon_\perp - n^2)(\mathbb{1} - \mathbf{hh}) + \varepsilon_\parallel \mathbf{hh} - \mathrm{i}g\mathbf{h} \times \mathbb{1}\right] \cdot \tilde{\mathbf{E}} = 0. \tag{2.138}$$

Of course, at this point we could just assume the systems of equation to be singular and solve for n^2. However, let's use a different approach. We multiply the equation (2.138) with $\mathbf{h}$ and obtain

$$\varepsilon_\parallel \mathbf{h} \cdot \tilde{\mathbf{E}} = 0. \tag{2.139}$$

So, for parallel propagation we either have $\varepsilon_\parallel = 0$ or $\mathbf{h} \cdot \tilde{\mathbf{E}} = 0$. Let's discuss them both.

- $\varepsilon_\parallel = 0$: In this case, the electric field perturbation must be parallel to the magnetic field $\tilde{\mathbf{E}} \parallel \mathbf{h}$. Also, the wave is electrostatic. Since $\tilde{\mathbf{E}} \propto \mathbf{k}$, we can write it in terms of the electrostatic potential,

  $$\tilde{\mathbf{E}} = -\mathrm{i}\mathbf{k}\tilde{\Phi}. \tag{2.140}$$

  Further, the curl of the electric field is

  $$\nabla \times \tilde{\mathbf{E}} \propto \tilde{\mathbf{B}} \tag{2.141}$$

  $$\mathbf{k} \times \tilde{\mathbf{E}} \propto \mathbf{k} \times \mathbf{k} = 0 \tag{2.142}$$

  $$\Rightarrow \nabla \times \tilde{\mathbf{E}} = 0, \tag{2.143}$$

  and hence, the electric field is static and so is the wave. In the cold plasma approximation, electrostatic waves can have any value of n [8] since (2.139) doesn't constrain it. We can still check what the equation $\varepsilon_\parallel = 0$ gives. Assuming a single ion species and an electron species, we have

  $$\varepsilon_\parallel = 1 - \sum_\sigma \frac{\omega_{p\sigma}^2}{\omega^2} \tag{2.144}$$

  $$= 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{Z_i m_e}{m_i}\right) \tag{2.145}$$

  $$\approx 1 - \frac{\omega_{pe}^2}{\omega^2} \tag{2.146}$$

  Hence, we have a wave with

  $$\omega \approx \omega_{pe} = \sqrt{\frac{4\pi e^2 n_e}{m_e}}. \tag{2.147}$$

This mode represents electrostatic oscillations along the magnetic field. These waves are called **Langmuir** waves. They pose charge oscillations along the magnetic field and are thus not constrained spatially (arbitrary k and n). Langmuir waves will occur again later when we consider hot electrostatic waves in section 2.3.2.

- $\mathbf{h} \cdot \tilde{\mathbf{E}} = 0$: In this case, which is not electrostatic, we have

$$\left[(\varepsilon_\perp - n^2)(\mathbb{1} - \mathbf{hh}) + \varepsilon_\parallel \mathbf{hh} - \mathrm{i}g\mathbf{h} \times \mathbb{1}\right] \cdot \tilde{\mathbf{E}} = 0 \tag{2.148}$$

$$(\varepsilon_\perp - n^2)\tilde{\mathbf{E}} - \mathrm{i}g\mathbf{h} \times \tilde{\mathbf{E}} = 0. \tag{2.149}$$

This can also be written in matrix vector notation as

$$\begin{pmatrix} \varepsilon_\perp - n^2 & \mathrm{i}g \\ -\mathrm{i}g & \varepsilon_\perp - n^2 \end{pmatrix} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \cdot \hat{\mathbf{x}} \\ \tilde{\mathbf{E}} \cdot \hat{\mathbf{y}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.150}$$

The determinant of this matrix is

$$\det \begin{pmatrix} \varepsilon_\perp - n^2 & \mathrm{i}g \\ -\mathrm{i}g & \varepsilon_\perp - n^2 \end{pmatrix} = (\varepsilon_\perp - n^2)^2 - g^2. \tag{2.151}$$

Setting this to zero, we get

$$n^2 = \varepsilon_\perp \pm g. \tag{2.152}$$

This solution corresponds to a polarization of the electric field perturbation that is

$$\tilde{\mathbf{E}} \propto \hat{\mathbf{x}} \mp \mathrm{i}\hat{\mathbf{y}}. \tag{2.153}$$

Since the temporal dependence of the wave is $\exp(-\mathrm{i}\omega t)$, the top sign corresponds to **left-handed circular polarization** and the bottom sign to **right-handed circular polarization**[5]. Figure 2.6 shows a sketch of the polarization.

Let's consider a plasma with only one ion species that has $Z_i = 1$, then

$$n^2 = \varepsilon_\perp \pm g \tag{2.154}$$

$$= 1 - \sum_\sigma \frac{\omega_{p\sigma}^2}{\omega^2 - \omega_{c\sigma}^2} \mp \sum_\sigma \frac{\omega_{p\sigma}^2 \omega_{c\sigma}}{\omega(\omega^2 - \omega_{c\sigma}^2)} \tag{2.155}$$

$$= 1 - \sum_\sigma \left[\frac{\omega_{p\sigma}^2(\omega \pm \omega_{c\sigma})}{\omega(\omega^2 - \omega_{c\sigma}^2)}\right] \tag{2.156}$$

$$= 1 - \sum_\sigma \frac{\omega_{p\sigma}^2}{\omega(\omega \mp \omega_{c\sigma})} \tag{2.157}$$

$$= 1 - \frac{\omega_{pi}^2}{\omega(\omega \mp \omega_{ci})} - \frac{\omega_{pe}^2}{\omega(\omega \pm |\omega_{ce}|)}. \tag{2.158}$$

[5]The polarization impacts how waves propagate through the plasma. Waves with different polarizations may experience different cutoffs (frequencies where the wave is reflected or absorbed) and resonances (frequencies where wave-particle interactions become efficient).

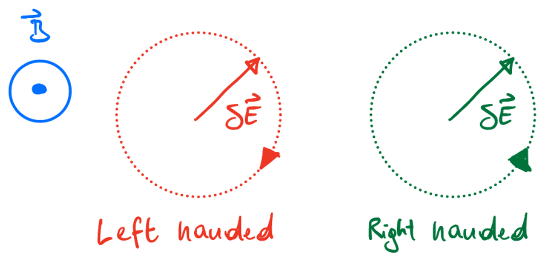

Figure 2.6: Sketch of the polarization of the electric field perturbation for parallel propagation.

Using the fact that

$$\frac{\omega_{pe}^2}{\omega_{pi}^2} = \frac{|\omega_{ce}|}{\omega_{ci}} \Rightarrow \omega_{pe}^2\omega_{ci} - \omega_{pi}^2|\omega_{ce}| = 0, \tag{2.159}$$

we continue with

$$\begin{aligned} n^2 &= 1 - \frac{\omega_{pi}^2\omega(\omega \pm |\omega_{ce}|) + \omega_{pe}^2\omega(\omega \mp \omega_{ci})}{\omega^2(\omega \mp \omega_{ci})(\omega \pm |\omega_{ce}|)} & (2.160)\\ &= \frac{\omega^2(\omega \mp \omega_{ci})(\omega \pm |\omega_{ce}|) - \omega_{pi}^2\omega(\omega \pm |\omega_{ce}|) - \omega_{pe}^2\omega(\omega \mp \omega_{ci})}{\omega^2(\omega \mp \omega_{ci})(\omega \pm |\omega_{ce}|)} & (2.161)\\ &\overset{(2.159)}{=} \frac{\omega^4 \pm \omega^3|\omega_{ce}| \mp \omega^3\omega_{ci} - \omega^2\omega_{ci}|\omega_{ce}| - \omega_{pi}^2\omega^2 - \omega_{pe}^2\omega^2}{\omega^2(\omega \mp \omega_{ci})(\omega \pm \omega_{ce})} & (2.162)\\ &= \frac{\omega^2 \pm (|\omega_{ce}| - \omega_{ci})\omega - \omega_{ci}|\omega_{ce}| - \omega_{pi}^2 - \omega_{pe}^2}{(\omega \mp \omega_{ci})(\omega \pm |\omega_{ce}|)}. & (2.163) \end{aligned}$$

The top sign still corresponds to left-handed polarization and the bottom sign to right-handed polarization. We introduce now the frequencies

$$\begin{aligned} \omega_L &= \sqrt{\left(\frac{|\omega_{ce}| + \omega_{ci}}{2}\right)^2 + \omega_{pe}^2 + \omega_{pi}^2} - \frac{|\omega_{ce}| - \omega_{ci}}{2}, & (2.164)\\ \omega_R &= \sqrt{\left(\frac{|\omega_{ce}| + \omega_{ci}}{2}\right)^2 + \omega_{pe}^2 + \omega_{pi}^2} + \frac{|\omega_{ce}| - \omega_{ci}}{2}. & (2.165) \end{aligned}$$

These definitions simplify the refraction index,

$$n^2 = \frac{(\omega \mp \omega_L)(\omega \pm \omega_R)}{(\omega \pm |\omega_{ce}|)(\omega \mp \omega_{ci})}. \tag{2.166}$$

The refraction index can show **cut-offs** and **resonances**. The former is a limit in the perturbation frequency ω above or below which the refraction index squared becomes negative, i.e. $n^2 < 0$. Hence, the wave becomes exponentially decaying and is called evanescent. Cut-offs are defined by the roots of the numerator. A resonance occurs for specific frequencies that result in $n^2 \to \infty$, i.e. at roots of the denominator. In this case, already a very small perturbation can lead to a significant plasma response. E.g. at the electron cyclotron resonance, the perturbation frequency of the wave matches the frequency of the cyclotron motion. This allows the particles to efficiently absorb the waves energy.

It is instructive to understand the ordering of the different frequencies ω_{pe}, ω_{pi}, ω_{ce}, ω_{ci}, ω_L and ω_R. Depending on the perturbation frequency ω the wave is of different type and will follow a different dispersion relation. We know that

$$\frac{\omega_{pi}^2}{\omega_{pe}^2} = \frac{\omega_{ci}}{|\omega_{ce}|} = \frac{Z_i m_e}{m_i} \ll 1. \tag{2.167}$$

The relative size of the frequencies is determined by the non-dimensional parameter $\omega_{pe}/|\omega_{ce}|$. In the most interesting cases in astrophysics and fusion physics, it holds that $\omega_{pe}/|\omega_{ce}| \gtrsim 1$, hence

$$\omega_L \simeq \sqrt{\frac{\omega_{ce}^2}{4} + \omega_{pe}^2} - \frac{|\omega_{ce}|}{2}, \tag{2.168}$$

$$\omega_R \simeq \sqrt{\frac{\omega_{ce}^2}{4} + \omega_{pe}^2} + \frac{|\omega_{ce}|}{2}. \tag{2.169}$$

This regime, implies an ordering of

$$\omega_R > \omega_{pe} > \omega_L \gg \omega_{ci}. \tag{2.170}$$

The frequency missing here is ω_{ce}. The relative size of this frequency in comparison to ω_{pe} and ω_L depends on the exact ratio of $\omega_{pe}/|\omega_{ce}|$. If we take as an example $\omega_{pe}/|\omega_{ce}| > \sqrt{2}$, we have

$$\omega_R > \omega_{pe} > \omega_L > |\omega_{ce}| \gg \omega_{ci}. \tag{2.171}$$

With (2.171) and (2.166), and the fact that n^2 has to be positive for wave propagation implies that the left-handed polarization exists if

$$\omega > \omega_L \quad \text{and} \quad \omega < \omega_{ci}, \tag{2.172}$$

while the right-handed polarization exists for

$$\omega > \omega_R \quad \text{and} \quad \omega < \omega_{ce}. \tag{2.173}$$

Note that the signs in (2.166) still correspond to left-handed (top sign) and right-handed (bottom sign) polarization and that the just discussed conditions for their existence stem from n^2 having to be positive.

Following the ordering (2.171), we find the following limits whose relation are sketched in figure 2.7:

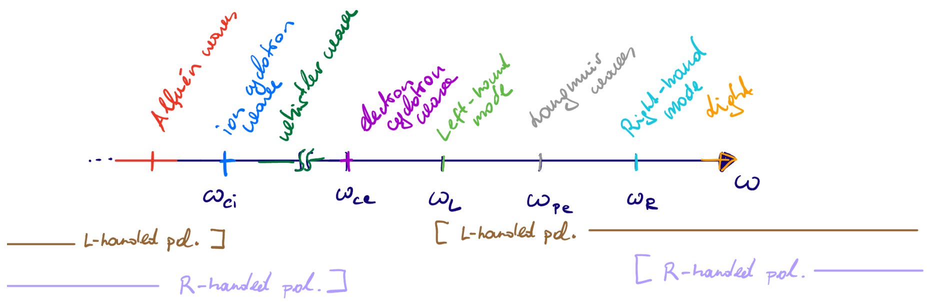

Figure 2.7: Sketch of the relation between the different waves depending on the perturbation frequency.

- For $\omega \gg \omega_R, \omega_L, |\omega_{ce}|, \omega_{ci}$, we find **light**

$$n^2 \approx 1 \Rightarrow \omega \approx kc. \tag{2.174}$$

It can have both left-handed and right-handed polarization.

- For $\omega \simeq |\omega_{ce}|$, we find the **electron cyclotron wave**. This wave has to be right-handed, as otherwise $\omega - \omega_L < 0$ which results in a negative n^2 in (2.166). We have

$$n^2 = \frac{(\omega \mp \omega_L)(\omega \pm \omega_R)}{(\omega \pm |\omega_{ce}|)(\omega \mp \omega_{ci})} \tag{2.175}$$

$$\overset{\omega \gg \omega_{ci}}{\approx} \frac{\omega^2 - \omega\omega_R + \omega_L\omega - \omega_L\omega_R}{\omega(\omega - |\omega_{ce}|)} \tag{2.176}$$

$$(\omega - |\omega_{ce}|)n^2 \simeq -\frac{\omega_L\omega_R}{\omega} \tag{2.177}$$

$$= -\frac{\omega_{pe}^2}{\omega}. \tag{2.178}$$

Where we have used that ω_L and ω_R are significantly greater than ω. From this (?), we arrive at

$$|\omega_{ce}| - \omega \simeq \frac{\omega_{pe}^2|\omega_{ce}|}{k^2c^2} \ll 1. \tag{2.179}$$

- For $|\omega_{ce}| \gg \omega \gg \omega_{ci}$, we have the **whistler wave**. This wave can only have right-handed polarization. Again, this is implied by the sign of n^2. In particular, for the left-handed case, $\omega + |\omega_{ce}| > 0$ is in the denominator, but the nominator has a negative sign due to $\omega - \omega_L < 0$. However, for the right-handed case, $\omega - \omega_R < 0$ in the nominator is negative but so is also $\omega - |\omega_{ce}|$ in the denominator and the sign cancels. Hence, we have

$$n^2 = \frac{(\omega \mp \omega_L)(\omega \pm \omega_R)}{(\omega \pm |\omega_{ce}|)(\omega \mp \omega_{ci})} \tag{2.180}$$

$$\approx \frac{\omega^2 + \omega_L\omega - \omega_R\omega - \omega_L\omega_R}{-|\omega_{ce}|\omega} \tag{2.181}$$

$$\approx \frac{\omega_L\omega_R}{|\omega_{ce}|\omega} \tag{2.182}$$

$$= \frac{\omega_{pe}^2}{|\omega_{ce}|\omega} \tag{2.183}$$

$$\Rightarrow \omega = \frac{|\omega_{ce}|c^2k^2}{\omega_{pe}^2}. \tag{2.184}$$

- For $\omega \simeq \omega_{ci}$, we have the **ion cyclotron wave**. This wave has left-handed polarization. Here, we have

$$n^2 = \frac{(\omega \mp \omega_L)(\omega \pm \omega_R)}{(\omega \pm |\omega_{ce}|)(\omega \mp \omega_{ci})} \tag{2.185}$$

$$\approx \frac{-\omega_L\omega_R}{|\omega_{ce}|(\omega - \omega_{ci})} \tag{2.186}$$

$$(\omega_{ci} - \omega)n^2 = \frac{\omega_{pe}^2}{|\omega_{ce}|} \tag{2.187}$$

$$= \frac{\omega_{pi}^2}{\omega_{ci}}. \tag{2.188}$$

Now, we introduce the Alfvén speed

$$v_A = \sqrt{\frac{B^2}{4\pi n_i m_i}} = c\frac{\omega_{ci}}{\omega_{pi}}. \tag{2.189}$$

With this definition, we have

$$(\omega_{ci} - \omega)n^2 = \frac{\omega_{ci}c^2}{v_A^2} \tag{2.190}$$

$$(\omega_{ci} - \omega) \simeq \frac{\omega_{ci}c^2}{v_A^2 \frac{c^2k^2}{\omega_{ci}^2}} \tag{2.191}$$

$$= \frac{\omega_{ci}^3}{v_A^2k^2} \ll 1. \tag{2.192}$$

- Finally, for $\omega_{ci}, |\omega_{ce}| \gg \omega$, we have Alfvén waves which can be either left-or right-handed. In this case,

$$n^2 = \frac{(\omega \mp \omega_L)(\omega \pm \omega_R)}{(\omega \pm |\omega_{ce}|)(\omega \mp \omega_{ci})} \tag{2.193}$$

$$\approx \frac{-\omega_L \omega_R}{-|\omega_{ce}|\omega_{ci}} \tag{2.194}$$

$$= \frac{\omega_{pe}^2}{|\omega_{ce}|\omega_{ci}} \tag{2.195}$$

$$= \frac{\omega_{pi}^2}{\omega_{ci}^2} \tag{2.196}$$

$$= \frac{c^2}{v_A^2} \Rightarrow \omega \simeq k v_A. \tag{2.197}$$

Perpendicular propagation

In the case that $\hat{\mathbf{k}} \cdot \mathbf{h} = 0$, the equation

$$\left[n^2(\hat{\mathbf{k}}\hat{\mathbf{k}} - \mathbb{1}) + \boldsymbol{\varepsilon} \right] \cdot \tilde{\mathbf{E}} = 0 \tag{2.198}$$

becomes

$$\left[\varepsilon_\perp(\mathbb{1} - \mathbf{hh}) - n^2(\mathbb{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}) + \varepsilon_\parallel \mathbf{hh} - \mathrm{i}g\mathbf{h} \times \mathbb{1} \right] \cdot \tilde{\mathbf{E}} = 0. \tag{2.199}$$

Multiplying this equation with $\mathbf{h}$ gives

$$(\varepsilon_\parallel - n^2)\mathbf{h} \cdot \tilde{\mathbf{E}} = 0. \tag{2.200}$$

This equation has two solutions: either $\varepsilon_\parallel - n^2 = 0$ or $\mathbf{h} \cdot \tilde{\mathbf{E}} = 0$. They are called **ordinary** and **extraordinary mode**.

Ordinary mode (O-mode): $\varepsilon_\parallel - n^2 = 0$ defines the phase velocity of the wave. For a plasma with a single ion species and electrons, we have

$$n^2 = \frac{k^2 c^2}{\omega^2} = \varepsilon_\parallel \tag{2.201}$$

$$= 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{Z_i m_e}{m_i}\right) \tag{2.202}$$

$$\Rightarrow \omega^2 \approx \omega_{pe}^2 + k^2 c^2. \tag{2.203}$$

The dispersion relation of the ordinary wave (O-wave) is the same as for an unmagnetized plasma, hence, the O-wave does not care about the magnetic field. Also, the wave has a lower cut-off at $\omega = \omega_{pe}$, i.e. frequencies below the plasma frequency are not allowed. The ordinary mode is linearly polarized and the electric field is parallel to the magnetic field $\tilde{\mathbf{E}} \parallel \mathbf{B}_0$. Therefore, the particles of the wave are not affected by the magnetic field.

The O-wave is used for plasma diagnostic to measure the plasma density. A wave is send through the plasma while a reference wave does not travel through the plasma (plasma interferometry). The phase difference between the two is a measure of the plasma density [10].

Extraordinary mode (X-mode): A wave that is elliptically polarized[6] and where the electric field has both parallel and perpendicular components to the magnetic field.

Assuming $\mathbf{h} \cdot \tilde{\mathbf{E}} = 0$, the equation (2.199) reads

$$\left[\varepsilon_\perp \mathbb{1} - n^2(\mathbb{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}) - \mathrm{i}g\mathbf{h} \times \mathbb{1}\right] \cdot \tilde{\mathbf{E}} = 0. \tag{2.204}$$

To solve this equation, we again project into the basis $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \mathbf{h})$, with $\hat{\mathbf{x}} = \mathbf{k}_\perp / k_\perp$, and $\hat{\mathbf{y}} = \mathbf{h} \times \hat{\mathbf{x}}$. In this case, we have

$$\begin{pmatrix} \varepsilon_\perp & \mathrm{i}g \\ -\mathrm{i}g & \varepsilon_\perp - n^2 \end{pmatrix} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \cdot \hat{\mathbf{x}} \\ \tilde{\mathbf{E}} \cdot \hat{\mathbf{y}} \end{pmatrix} = 0. \tag{2.205}$$

The polarization of the electric field is

$$\tilde{\mathbf{E}} \propto \mathrm{i}g\hat{\mathbf{x}} - \varepsilon_\perp \hat{\mathbf{y}}. \tag{2.206}$$

Since the relative magnitude in $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions vary, this wave has elliptical polarization.

The determinant of the matrix has to be zero, hence

$$n^2 = \varepsilon_\perp - \frac{g^2}{\varepsilon_\perp} \tag{2.207}$$

$$= \frac{\varepsilon_\perp^2 - g^2}{\varepsilon_\perp}. \tag{2.208}$$

We determined earlier (equation (2.166)) that

$$\varepsilon_\perp \pm g = \frac{(\omega \mp \omega_L)(\omega \pm \omega_R)}{(\omega \pm \omega_{ce})(\omega \mp \omega_{ci})}. \tag{2.209}$$

Hence,

$$\varepsilon_\perp^2 - g^2 = (\varepsilon_\perp + g)(\varepsilon_\perp - g) \tag{2.210}$$

$$= \frac{(\omega^2 - \omega_L^2)(\omega^2 - \omega_R^2)}{(\omega^2 - \omega_{ce}^2)(\omega^2 - \omega_{ci}^2)}. \tag{2.211}$$

Also,

$$\varepsilon_\perp = 1 - \sum_\sigma \frac{\omega_{p\sigma}^2}{\omega^2 - \omega_{c\sigma}^2} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2}. \tag{2.212}$$

[6] An elliptical polarization is a superposition of two linearly polarized waves where either the amplitudes are different, or the phase difference between the waves varies. If the phase difference is 90°, the polarization is circular.

This can be inserted back into (2.208), and using $\omega_{pi}^2\omega_{ce} = \omega_{pe}^2\omega_{ci}$, we have

$$n^2 = \frac{(\omega^2 - \omega_L^2)(\omega^2 - \omega_R^2)}{\omega^4 - (\omega_{ce}^2 + \omega_{ci}^2 + \omega_{pe}^2 + \omega_{pi}^2)\omega^2 + \omega_{ce}^2\omega_{ci}^2 + \omega_{pe}^2\omega_{ci}(\omega_{ce} + \omega_{ci})}. \tag{2.213}$$

This expression can be simplified by introducing the **upper hybrid frequency**

$$\omega_{UH}^2 = \frac{\omega_{ce}^2 + \omega_{ci}^2 + \omega_{pe}^2 + \omega_{pi}^2}{2} + \sqrt{\left(\frac{\omega_{ce}^2 + \omega_{ci}^2 + \omega_{pe}^2 + \omega_{pi}^2}{2}\right)^2 - \omega_{ce}^2\omega_{ci}^2 - \omega_{pe}^2\omega_{ci}(\omega_{ce} + \omega_{ci})} \tag{2.214}$$

and the **lower hybrid frequency**

$$\omega_{LH}^2 = \frac{\omega_{ce}^2 + \omega_{ci}^2 + \omega_{pe}^2 + \omega_{pi}^2}{2} - \sqrt{\left(\frac{\omega_{ce}^2 + \omega_{ci}^2 + \omega_{pe}^2 + \omega_{pi}^2}{2}\right)^2 - \omega_{ce}^2\omega_{ci}^2 - \omega_{pe}^2\omega_{ci}(\omega_{ce} + \omega_{ci})}. \tag{2.215}$$

Then, we have

$$n^2 = \frac{(\omega^2 - \omega_L^2)(\omega^2 - \omega_R^2)}{(\omega^2 - \omega_{UH}^2)(\omega^2 - \omega_{LH}^2)}. \tag{2.216}$$

Using $\omega_{pi}^2/\omega_{pe}^2 = \omega_{ci}/\omega_{ce} = Zm_e/m_i \ll 1$, we can again infer on the relative size of ω_{pe}, ω_R, ω_L, ω_{LH} and ω_{UH} which is determined by the non-dimensional ratio ω_{pe}/ω_{ce}. For $\omega_{pe}/\omega_{ce} \gtrsim 1$, we have

$$\omega_{UH} \simeq \sqrt{\omega_{pe}^2 + \omega_{ce}^2}, \tag{2.217}$$

$$\omega_{LH} \simeq \sqrt{\frac{\omega_{ci}\omega_{ce}}{1 + \omega_{ce}^2/\omega_{pe}^2}}. \tag{2.218}$$

Therefore,

$$\omega_R > \omega_{UH} > \omega_{pe} > \omega_L \gg \omega_{LH}. \tag{2.219}$$

According to (2.213), there are extraordinary waves for $\omega > \omega_R$, $\omega_L < \omega < \omega_{UH}$, and $\omega < \omega_{LH}$.

Clearly, equation (2.216) shows two resonances $\omega = \pm\omega_{UH}$ and $\omega = \pm\omega_{LH}$ which are consequently called upper and lower hybrid resonances.

2.3.2 Finite temperature effects in electrostatic wave dispersion

We continue by considering finite temperature effects in linear plasma waves. The simplest type of wave where these effects manifest is Langmuir waves, i.e. electrostatic waves. To study Langmuir waves, it is sufficient to consider an unmagnetized plasma ($\mathbf{B}_0 = 0$). Further, we assume no equilibrium electric field ($\mathbf{E}_0 = 0$) and that the perturbation is in $\hat{x}$ direction,

$$\delta\mathbf{E}(\mathbf{x}) = \widetilde{E}\hat{\mathbf{x}}\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{2.220}$$

For the purpose of this section we are also only interested in longitudinal waves, i.e. $\hat{\mathbf{k}} = \hat{\mathbf{x}}$, hence,

$$\delta\mathbf{E}(\mathbf{x}) = \widetilde{E}\hat{\mathbf{x}}\mathrm{e}^{\mathrm{i}(kx-\omega t)}. \tag{2.221}$$

Note that the assumption that $\widetilde{\mathbf{E}} = \widetilde{E}\hat{\mathbf{x}} \parallel \hat{\mathbf{k}}$ implies that the electric field is electrostatic, i.e., there is no magnetic field perturbation, since from Faraday's law

$$0 = \mathbf{k} \times \widetilde{\mathbf{E}} = -\frac{\omega}{c}\widetilde{\mathbf{B}}. \tag{2.222}$$

Given the assumptions, we treat this problem as one-dimensional. In the following, we derive the dispersion equation for this case without restricting ourselves to a cold plasma.

For the given scenario, the Vlasov equation for each species σ is

$$\frac{\partial f_\sigma}{\partial t} + v_x\frac{\partial f_\sigma}{\partial x} + \frac{eZ_\sigma}{m_\sigma}\delta E\frac{\partial f_\sigma}{\partial v_x} = 0. \tag{2.223}$$

Here, we already omitted terms of the gradients that vanish due to the assumptions we made, viz. that the wave propagates in $\hat{\mathbf{x}}$ direction and that the electric field perturbation is in the same direction. We linearize the distribution function,

$$f_\sigma(\mathbf{x}, \mathbf{v}, t) = f_{\sigma 0}(\mathbf{v}) + \delta f_\sigma(\mathbf{x}, \mathbf{v}, t), \tag{2.224}$$

where we assume that the background is uniform in space (no gradients) and time (steady-state)[7]. We will further write the perturbation as a monochromatic plane wave,

$$\delta f_\sigma(\mathbf{x}, \mathbf{v}, t) = \tilde{f}_\sigma(\mathbf{v})\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{2.225}$$

$$= \tilde{f}_\sigma(\mathbf{v})\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{2.226}$$

since $\mathbf{k} \parallel \hat{\mathbf{x}}$. The equilibrium part of the distribution function trivially satisfies

$$\frac{\partial f_{\sigma 0}(\mathbf{v})}{\partial t} + v_x\frac{\partial f_{\sigma 0}(\mathbf{v})}{\partial x} = 0. \tag{2.227}$$

Hence, the perturbation of the distribution function is governed by

$$\frac{\partial \delta f_\sigma}{\partial t} + v_x\frac{\partial \delta f_\sigma}{\partial x} + \frac{eZ_\sigma}{m_\sigma}\delta E\frac{\partial f_{\sigma 0}}{\partial v_x} = 0, \tag{2.228}$$

where we neglected terms of higher order in the perturbation (linearization).

[7] Note that the equilibrium can be written as a function of the velocity components since the momenta in each direction are conserved, c.f. section 2.2.1.

Now, applying the monochromatic plane wave ansatz to solve for δf_σ, the Vlasov equation becomes algebraic,

$$-\mathrm{i}\omega\tilde{f}_\sigma + \mathrm{i}kv_x\tilde{f}_\sigma = -\frac{eZ_\sigma}{m_\sigma}\widetilde{E}\frac{\partial f_{\sigma 0}}{\partial v_x}, \tag{2.229}$$

which is solved to be

$$\tilde{f}_\sigma = \frac{-\mathrm{i}eZ_\sigma}{m_\sigma(\omega - kv_x)}\widetilde{E}\frac{\partial f_{\sigma 0}}{\partial v_x}. \tag{2.230}$$

We see that the distribution function perturbation depends on the velocity derivative of the equilibrium distribution function and is driven by the electric field perturbation.

To get the dispersion equation, we need to consider Gauss' law,

$$\nabla \cdot \delta\mathbf{E} = 4\pi\delta\rho. \tag{2.231}$$

Recall, that the charge density perturbation is determined by the zeroth moment of the distribution function perturbation. Thus, with the wave ansatz, we can write Gauss' equation as

$$\mathrm{i}k\widetilde{E} = 4\pi e\int \mathrm{d}^3v\,\left(Z_i\tilde{f}_i - \tilde{f}_e\right), \tag{2.232}$$

for a simple plasma with one ion species with charge number Z_i.

Inserting the solution to Vlasov's equation we have

$$\mathrm{i}k\widetilde{E} = -\mathrm{i}4\pi e^2\widetilde{E}\int \mathrm{d}^3v\,\left(\frac{Z_i^2}{m_i}\frac{\partial_{v_x}f_{i0}}{\omega - kv_x} + \frac{1}{m_e}\frac{\partial_{v_x}f_{e0}}{\omega - kv_x}\right) \tag{2.233}$$

$$\mathrm{i}k\left(1 + 4\pi e^2\int \mathrm{d}^3v\,\left(\frac{Z_i^2}{m_i}\frac{\partial_{v_x}f_{i0}}{\omega - kv_x} + \frac{1}{m_e}\frac{\partial_{v_x}f_{e0}}{\omega - kv_x}\right)\right)\widetilde{E} = 0 \tag{2.234}$$

$$\mathrm{i}k\varepsilon(k,\omega)\widetilde{E} = 0, \tag{2.235}$$

where we defined the dielectric function

$$\varepsilon(k,\omega) = 1 + 4\pi e^2\int \mathrm{d}^3v\,\left(\frac{Z_i^2}{m_i}\frac{\partial_{v_x}f_{i0}}{\omega - kv_x} + \frac{1}{m_e}\frac{\partial_{v_x}f_{e0}}{\omega - kv_x}\right) \tag{2.236}$$

$$= 1 + \frac{4\pi e^2 n_e}{m_e}\int \mathrm{d}v_x\,\left(\frac{Z_i^2 m_e}{m_i n_e}\frac{\int \mathrm{d}v_y\mathrm{d}v_z\,\partial_{v_x}f_{i0}}{\omega - kv_x} + \frac{1}{n_e}\frac{\int \mathrm{d}v_y\mathrm{d}v_z\,\partial_{v_x}f_{e0}}{\omega - kv_x}\right). \tag{2.237}$$

The prefactor $4\pi e^2 n_e/m_e = \omega_{pe}^2$ equals the electron plasma frequency (squared). Further, we define a function

$$g(v_x) = \frac{Z_i^2 m_e}{n_e m_i}\iint \mathrm{d}v_y\mathrm{d}v_z\,f_{i0}(\mathbf{v}) + \frac{1}{n_e}\iint \mathrm{d}v_y\mathrm{d}v_z\,f_{e0}(\mathbf{v}). \tag{2.238}$$

With this, and pulling out a factor k from the denominator, we arrive at the dielectric function

Electrostatic dielectric function

$$\varepsilon(k,\omega) = 1 + \frac{\omega_{pe}^2}{k^2}\int \mathrm{d}v_x\, \frac{\partial_{v_x} g(v_x)}{\omega/k - v_x}. \tag{2.239}$$

The first term in ε is due to the vacuum part of Gauss' law, i.e. the right hand side. The second term incorporates the response of the plasma. Hence, it contains the properties of the plasma equilibrium.

The equation deduced from Gauss' law, equation (2.235), has two solutions. First, the trivial one, $\widetilde{E} = 0$, which is not interesting. Second, setting $\varepsilon(k,\omega) = 0$. Setting the dielectric function to zero, this equation can be solved for the dispersion relation $\omega(k)$, depending on the equilibrium distribution function of the species.

Note that in $g(v_x)$ the contribution of the ions is a factor m_e/m_i smaller than that of the electrons. For example, in a deuterium-electron plasma, this would be a factor of about 4000. However, considering aloso the integral over the distribution function, this ratio is weakened to $\sqrt{m_e/m_i} \approx 1/100$ which is still a large factor.

Furthermore, notice the denominator of the integrand, $\omega/k - v_x$. It is the difference between the phase velocity of the wave, $v_p = \omega/k$, and the particle velocity v_x and shows a root where they two are equal. Hence, the integrand has a singularity. How to handle this pole is an extensive topic in itself [7]. For simplicity, in this lecture, we will avoid treating it rigorously. Nevertheless, we will see that this resonance condition plays a significant role in Landau damping 2.3.4.

Let's determine $g(v_x)$ for the case that the ions and electrons follow a Maxwellian distribution in equilibrium, i.e.

$$f_{i0}(\mathbf{v}) = \frac{n_i}{(2\pi v_{Ti}^2)^{3/2}} \exp\left(-\frac{v_x^2+v_y^2+v_z^2}{2v_{Ti}^2}\right) \tag{2.240}$$

$$f_{e0}(\mathbf{v}) = \frac{n_e}{(2\pi v_{Te}^2)^{3/2}} \exp\left(-\frac{v_x^2+v_y^2+v_z^2}{2v_{Te}^2}\right). \tag{2.241}$$

Then,

$$\begin{aligned} g(v_x) &= \frac{Z_i^2 m_e}{n_e m_i}\int \mathrm{d}v_y \mathrm{d}v_z\, f_{i0}(\mathbf{v}) + \frac{1}{n_e}\int \mathrm{d}v_y \mathrm{d}v_z\, f_{e0}(\mathbf{v}) \\ &= \frac{Z_i^2 m_e}{n_e m_i} 2\pi v_{Ti}^2 \frac{n_i}{(2\pi v_{Ti}^2)^{3/2}} \mathrm{e}^{-\frac{v_x^2}{2v_{Ti}^2}} + \frac{1}{n_e} 2\pi v_{Te}^2 \frac{n_e}{(2\pi v_{Te}^2)^{3/2}} \mathrm{e}^{-\frac{v_x^2}{2v_{Te}^2}} \qquad (2.242) \\ &= Z_i \frac{m_e}{m_i} \frac{1}{\sqrt{2\pi} v_{Ti}} \mathrm{e}^{-\frac{v_x^2}{2v_{Ti}^2}} + \frac{1}{\sqrt{2\pi} v_{Te}} \mathrm{e}^{-\frac{v_x^2}{2v_{Te}^2}}. \qquad (2.243) \end{aligned}$$

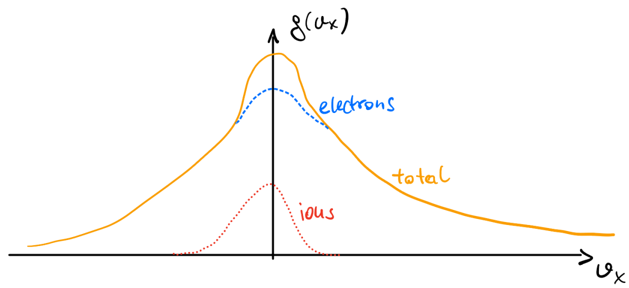

Figure 2.8: Plot of $g(v_x)$ for $T_i = T_e$.

Here, we used the Gaussian integral $\int_{-\infty}^{\infty} \mathrm{d}x \exp(-x^2 a) = \sqrt{\pi/a}$ and the quasineutrality condition $\sum_\sigma Z_\sigma n_\sigma = -n_e + Z_i n_i = 0$. For equal temperature, $T_i = T_e$, the thermal velocites are $v_{Ti} \ll v_{Te}$, meaning, that the ion contribution to g falls off way more rapidly than the electron contribution. Furthermore, as mentioned before, the contribution of the ions is a factor $m_e/m_i \ll 1$ smaller than that of electrons. A sketch of this behavior is shown in figure 2.8.

Let's first solve the dispersion equation (2.239) for a limiting case.

2.3.3 Warm Langmuir waves

Let's focus here on high frequency electron waves, i.e. Langmuir waves. We encountered them already in the cold plasma wave section 2.3.1. In the following, in contrast to the cold plasma, we will determine a correction to the dispersion relation due to the thermal motion of the plasma.

Assuming a high perturbation frequency implies that the heavy ions are too slow to react and we can neglect their contribution. Further, for now, let's put our attention to waves with large phase velocity in the sense that $\omega/k \gg v_x$ for any non-zero value of $g(v_x)$ (sketch in figure 2.9)[8]. This is called **warm plasma** assumption. Using this assumption, we can apply partial integration and Taylor series expansion in (2.239) and skip the pole of the integrand at $v_x = \omega/k$.

Let's start with the partial integration in the dielectric function

$$\varepsilon(k,\omega) = 1 + \frac{\omega_{pe}^2}{k^2} \int \mathrm{d}v_x \, \frac{1}{\omega/k - v_x} \partial_{v_x} g(v_x) \tag{2.244}$$

$$= 1 + \frac{\omega_{pe}^2}{k^2} \left[\frac{g(v_x)}{\omega/k - v_x} \right]_{v_x=-\infty}^{v_x=\infty} - \frac{\omega_{pe}^2}{k^2} \int \mathrm{d}v_x \, g(v_x) \partial_{v_x} \left(\frac{1}{\omega/k - v_x} \right) \tag{2.245}$$

$$= 1 - \frac{\omega_{pe}^2}{k^2} \int \mathrm{d}v_x \, g(v_x) \frac{1}{(\omega/k - v_x)^2}. \tag{2.246}$$

[8] Although this seems deceptively similar to the cold plasma case, we make this assumption after we have determined the dispersion equation.

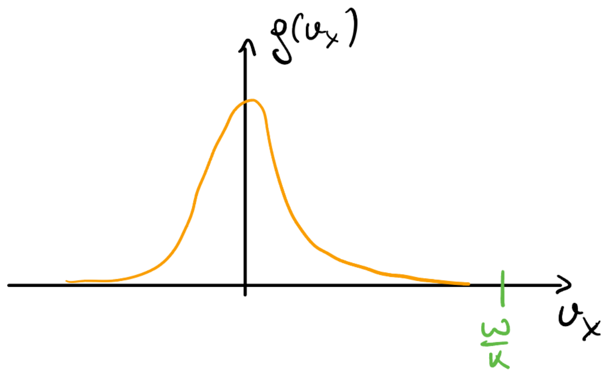

Figure 2.9: Sketch of $g(v_x)$ for which ω/k is large for any non-zero value of g.

We can Taylor expand the denominator of the integrand

$$
\begin{aligned}
\frac{1}{(\omega/k - v_x)^2} &= \frac{k^2}{\omega^2}\frac{1}{(1-kv_x/\omega)^2} \\
&= \frac{k^2}{\omega^2}\left(1 + 2\frac{kv_x}{\omega} + 3\frac{k^2v_x^2}{\omega^2} + \mathcal{O}\left(\frac{k^3v_x^3}{\omega^3}\right)\right). \qquad (2.247)
\end{aligned}
$$

We will see in a minute while we include the second order (v_x^2) in our consideration. Using this in the integral, we get

$$
\begin{aligned}
\varepsilon(k,\omega) &= 1 - \frac{\omega_{pe}^2}{\omega^2}\int \mathrm{d}v_x\ g(v_x)\frac{1}{(\omega/k - v_x)^2} && (2.248)\\
&= 1 - \frac{\omega_{pe}^2}{\omega^2}\int \mathrm{d}v_x\ g(v_x)\left(1 + 2\frac{kv_x}{\omega} + 3\frac{k^2v_x^2}{\omega^2} + \mathcal{O}\left(\frac{k^3v_x^3}{\omega^3}\right)\right) && (2.249)\\
&= 1 - \frac{\omega_{pe}^2}{\omega^2}\int \mathrm{d}v_x\ g(v_x) - 2\frac{\omega_{pe}^2 k}{\omega^3}\int \mathrm{d}v_x\ g(v_x)v_x && (2.250)\\
&- 3\frac{\omega_{pe}^2k^2}{\omega^4}\int \mathrm{d}v_x\ g(v_x)v_x^2 + \mathcal{O}\left(\frac{k^3}{\omega^5}\right) && (2.251)\\
&\overset{(2.243)}{=} 1 - \frac{\omega_{pe}^2}{\omega^2}\frac{1}{\sqrt{2\pi}v_{Te}}\int \mathrm{d}v_x\ \exp\left(-\frac{v_x^2}{2v_{Te}^2}\right) && (2.252)\\
&- 2\frac{\omega_{pe}^2k}{\omega^3}\frac{1}{\sqrt{2\pi}v_{Te}}\int \mathrm{d}v_x\ \exp\left(-\frac{v_x^2}{2v_{Te}^2}\right)v_x && (2.253)\\
&- 3\frac{\omega_{pe}^2k^2}{\omega^4}\frac{1}{\sqrt{2\pi}v_{Te}}\int \mathrm{d}v_x\ \exp\left(-\frac{v_x^2}{2v_{Te}^2}\right)v_x^2 + \mathcal{O}\left(\frac{k^3}{\omega^5}\right). && (2.254)
\end{aligned}
$$

The Gauss integral (2.252) is $\sqrt{2\pi}v_{Te}$, the integral (2.253) is zero (odd function over symmetric interval), and the integral (2.254) is $v_{Te}^3\sqrt{2\pi}$ which is due to $\int \mathrm{d}x\ x^2\exp(-x^2/a) = \sqrt{\pi/a}/(2a)$. Since the second integral is zero, we had to include the term of order v_x^2 to get a next order effect.

Hence, we have

$$1 - \frac{\omega_{pe}^2}{\omega^2} \frac{1}{\cancel{\sqrt{2\pi} v_{Te}}} \cancel{\sqrt{2\pi} v_{Te}} - 3\frac{\omega_{pe}^2 k^2}{\omega^4} \frac{1}{\cancel{\sqrt{2\pi} v_{Te}}} \frac{\cancel{2} v_{Te}^2}{\cancel{2}} \cancel{\sqrt{2\pi v_{Te}^2}} \tag{2.255}$$

$$= 1 - \frac{\omega_{pe}^2}{\omega^2} - 3\frac{\omega_{pe}^2 k^2}{\omega^4} v_{Te}^2 = 0, \tag{2.256}$$

which gives us

$$\omega^4 - \omega_{pe}^2 \omega^2 - 3\omega_{pe}^2 k^2 v_{Te}^2 = 0. \tag{2.257}$$

This is a biquadratic equation for ω. Using the quadratic formula, we get

$$\omega^2 = \frac{1}{2}\left(\omega_{pe}^2 \pm \sqrt{\omega_{pe}^4 + 12\omega_{pe}^2 k^2 v_{Te}^2}\right) \tag{2.258}$$

$$= \frac{\omega_{pe}^2}{2}\left(1 \pm \sqrt{1 + 12\frac{k^2 v_{Te}^2}{\omega_{pe}^2}}\right) \tag{2.259}$$

$$\approx \frac{\omega_{pe}^2}{2}\left(1 \pm 1 + 6\frac{k^2 v_{Te}^2}{\omega_{pe}^2}\right) \tag{2.260}$$

$$= \frac{\omega_{pe}^2}{2} \pm \frac{\omega_{pe}^2}{2} + 3k^2 v_{Te}^2. \tag{2.261}$$

Here, we want to choose the upper sign, since we are interested in high frequency oscillations, and typically $\omega_{pe}^2 \gg k^2 v_{Te}^2$. Finally, this gives us the

Warm Langmuir wave dispersion relation

$$\omega(k)^2 = \omega_{pe}^2 + 3k^2 v_{Te}^2, \tag{2.262}$$

which is also called the **Bohm-Gross** dispersion relation. The second term on the right is a modification due to the **finite temperature** of the plasma, but still only in the warm plasma limit. In a cold plasma, where $\omega/k \gg v_{Te}$ is used from the outset, this term would be absent and we would have gotten simple electron plasma oscillations with $\omega^2 = \omega_{pe}^2$.

2.3.4 Landau damping

Now, we relax the restriction of warm plasma, i.e. the phase velocity is well within the reach of the particle population. We will see that a collisionless damping effect occurs, which is called Landau damping. Most notably, Landau damping is a process where waves are damped in a collisionless plasma by resonant interaction with the particles. This process happens without any collisional dissipation. Energy is redistributed from the wave to the particles.

The process of Landau damping is not only an important topic in plasma physics, but it also plays a role in the dynamics of galaxies where gravity substitutes the long-range Coulomb interaction. The damping has a stabilizing effect on the galaxy dynamic. In plasma physics, Landau damping occurs for all types of modes. However, we discuss it only for the case of Langmuir waves, which is the simplest one.

The significance of Landau damping is that a time-reversible [9] system of equations (Vlasov-Poisson) describes a time-irreversable effect (dissipative damping). In essence, energy dissipates from the wave to the particles. This effect is similar to mechanical friction where for example a car on the road dissipates kinetic energy by the frictional contact to the road.

Here, we will only treat the linear version of Landau damping. However, keep in mind that some scenarios require a more rigorous non-linear treatment (e.g. for trapped particles [10]).

The mathematically rigorous way to treat this collisionless damping effect was first shown by Landau [6] by posing the problem in terms of an initial-value problem. Landau then used Laplace and Fourier transforms to solve the problem. This treatment includes the correct handling of the poles occurring in the dielectric function by means of complex integration and the Landau contour. In the following, we discuss Landau damping in a simplified way.

We still consider the same plasma scenario as in the previous section, i.e., an unmagnetized single ion-species plasma without collisions that is perturbed by an electric field. Also, we neglect ion dynamics since they are significantly heavier than the electrons. The perturbation is given by a time-harmonic plane wave and is considered small. The dispersion equation upon linearization of Vlasov's equation is given by (2.239).

Recalling that the wave is described by $\propto \exp(-\mathrm{i}\omega t)$ we see that if the frequency has an imaginary contribution, $\omega = \omega_r + \mathrm{i}\omega_i$, the wave is either increasing or damped exponentially over time. Under the assumption of a complex frequency, we determine its real and imaginary part.

To calculate the real and imaginary part of ω, we assume that $|\omega_i| \ll |\omega_r|$, which we will show later. First, we split the dielectric function into real and imaginary part

$$\varepsilon(k,\omega) = 1 + \frac{\omega_{pe}^2}{k^2}\int \mathrm{d}u\, \frac{1}{\omega/k - u}\frac{\partial g(u)}{\partial u} \tag{2.263}$$

$$= \varepsilon_r(k,\omega) + \mathrm{i}\varepsilon_i(k,\omega). \tag{2.264}$$

Using the assumption that the imaginary part of the frequency is smaller than the real part, we Taylor expand the real and imaginary part of the dielectric

[9] Time-reversible means that the equations of motion are symmetric under $t \to -t$ and $v \to -v$.

function

$$\varepsilon(k,\omega) = \varepsilon_r(k,\omega_r) + \mathrm{i}\varepsilon_i(k,\omega_r) + (\omega-\omega_r)\frac{\partial\varepsilon_r(k,\omega)}{\partial\omega}\bigg|_{\omega=\omega_r} \tag{2.265}$$

$$+ (\omega-\omega_r)\frac{\partial\varepsilon_i(k,\omega)}{\partial\omega}\bigg|_{\omega=\omega_r} + \mathcal{O}(\omega_i^2) \tag{2.266}$$

$$\approx \varepsilon_r(k,\omega_r) + \mathrm{i}\varepsilon_i(k,\omega_r) + \mathrm{i}\omega_i\frac{\partial\varepsilon_r(k,\omega)}{\partial\omega}\bigg|_{\omega=\omega_r}, \tag{2.267}$$

$$= \varepsilon_r(k,\omega_r) + \mathrm{i}\left(\varepsilon_i(k,\omega_r) + \omega_i\frac{\partial\varepsilon_r(k,\omega)}{\partial\omega}\bigg|_{\omega=\omega_r}\right) \tag{2.268}$$

where we neglect the derivative of ε_i because it is a product of ω_i and $\varepsilon_i \sim \omega_i$.

Setting this equation to zero to get the eigenmode solutions (dispersion relation), both the real part and the imaginary part have to vanish independently. That is,

$$\varepsilon_r(k,\omega_r) = 0, \tag{2.269}$$

which determines ω_r, and

$$0 = \varepsilon_i + \omega_i\frac{\partial\varepsilon_r}{\partial\omega}\bigg|_{\omega=\omega_r} \Rightarrow \omega_i = -\frac{\varepsilon_i(k,\omega_r)}{\partial\varepsilon_r/\partial\omega|_{\omega=\omega_r}}, \tag{2.270}$$

which determines ω_i.

To get ω_r and ω_i, we need to figure out what ε_r and ε_i are. To do so, we go back to the definition of the dielectric function

$$\varepsilon(k,\omega) = 1 - \frac{\omega_{pe}^2}{k^2}\int \mathrm{d}u\ \frac{1}{u-\omega_r/k-\mathrm{i}\omega_i/k}\frac{\partial g(u)}{\partial u}. \tag{2.271}$$

This integral can be rewritten using the Sokhotski-Plemelj theorem from complex analysis,

$$\lim_{b\to 0}\frac{1}{u-a\pm\mathrm{i}|b|} = P\left(\frac{1}{u-a}\right) \mp \mathrm{i}\pi\delta(u-a). \tag{2.272}$$

Here, P is the principal value given by

$$P\int_{-\infty}^{\infty}\mathrm{d}u\ \frac{f(u)}{u-a} = \lim_{\epsilon\to 0^+}\left(\int_{-\infty}^{a-\epsilon}\mathrm{d}u\frac{f(u)}{u-a} + \int_{a+\epsilon}^{\infty}\mathrm{d}u\ \frac{f(u)}{u-a}\right), \tag{2.273}$$

where $f(u)$ is any function. We can use the Sokhotski-Plemelj theorem to rewrite the dielectric function

$$\varepsilon(k,\omega) = 1 - \frac{\omega_{pe}^2}{k^2}\int \mathrm{d}u \, \frac{1}{u - \omega_r/k - \mathrm{i}\omega_i/k}\frac{\partial g(u)}{\partial u} \tag{2.274}$$

$$= 1 - \frac{\omega_{pe}^2}{k^2} P \int \mathrm{d}u \left(\frac{1}{u - \omega_r/k}\frac{\partial g(u)}{\partial u} - \mathrm{i}\pi\delta(u - \omega_r/k)\frac{\partial g(u)}{\partial u} \right) \tag{2.275}$$

$$= \underbrace{1 - \frac{\omega_{pe}^2}{k^2} P \int \mathrm{d}u \, \frac{1}{u - \omega_r/k}\frac{\partial g(u)}{\partial u}}_{\varepsilon_r} + \underbrace{\left(-\mathrm{i}\frac{\omega_{pe}^2 \pi}{k^2}\frac{\partial g(u)}{\partial u}\bigg|_{u=\omega_r/k} \right)}_{\varepsilon_i}. \tag{2.276}$$

This gives us now expressions for the real and imaginary parts of ε. We can solve the real part as before by partial integration and by assuming that $\partial g(u)/\partial u = 0$ at the pole since we are not interested in the details of this integration. Then

$$\varepsilon_r(k,\omega_r) = 1 - \frac{\omega_{pe}^2}{k^2} P \int \mathrm{d}u \, \frac{1}{u - \omega_r/k}\frac{\partial g(u)}{\partial u} = 0 \tag{2.277}$$

$$\Rightarrow \varepsilon_r = 1 - \frac{\omega_{pe}^2}{\omega_r^2} - \frac{3k^2 v_{Te}^2 \omega_{pe}^2}{\omega_r^4} \tag{2.278}$$

$$\Rightarrow \omega_r^2 = \omega_{pe}^2 + 3k^2 v_{Te}^2 \tag{2.279}$$

$$\omega_r \approx \omega_{pe}(1 + \frac{3}{2}k^2\lambda_D^2), \tag{2.280}$$

which is the Bohm-Gross dispersion relation.

More interestingly, we can evaluate the imaginary part of the frequency,

$$\omega_i = -\frac{\varepsilon_i(k,\omega_r)}{\partial\varepsilon_r/\partial\omega|_{\omega=\omega_r}} \tag{2.281}$$

$$= \frac{1}{\partial\varepsilon_r/\partial\omega|_{\omega=\omega_r}} \frac{\omega_{pe}^2\pi}{k^2}\frac{\partial g(u)}{\partial u}\bigg|_{u=\omega_r/k} \tag{2.282}$$

$$= \left(\frac{2\omega_{pe}^2}{\omega_r^2} + \frac{13k^2 v_{Te}^2 \omega_{pe}^2}{\omega_r^5} \right)^{-1} \frac{\omega_{pe}^2\pi}{k^2}\frac{\partial g(u)}{\partial u}\bigg|_{u=\omega_r/k} \tag{2.283}$$

$$\approx \left(\frac{2\omega_{pe}^2}{\omega_r^2} \right)^{-1} \frac{\omega_{pe}^2\pi}{k^2}\frac{\partial g(u)}{\partial u}\bigg|_{u=\omega_r/k} \tag{2.284}$$

$$\approx \left(\frac{2}{\omega_{pe}} \right)^{-1} \frac{\omega_{pe}^2\pi}{k^2}\frac{\partial g(u)}{\partial u}\bigg|_{u=\omega_r/k} \tag{2.285}$$

$$\Rightarrow \omega_i = \frac{\omega_{pe}^3\pi}{2k^2}\frac{\partial g(u)}{\partial u}\bigg|_{u=\omega_r/k}. \tag{2.286}$$

Finally, we have the angular frequency as a function of the wave number

Dispersion relation with Landau damping or growth

$$\omega(k) = \omega_{pe}(1 + 3k^2\lambda_D^2) + \mathrm{i}\frac{\omega_{pe}^3\pi}{2k^2}\frac{\partial g(u)}{\partial u}\bigg|_{u=\omega_r/k}. \tag{2.287}$$

This equation is valid for Langmuir waves such that $\lambda_D k \ll 1$, i.e. the wavelength of the wave is large in comparison to the Debye length.

Clearly, the slope of the distribution function $\partial g(u)/\partial u|_{u=\omega_r/k}$ determines if a wave with phase velocity ω_r/k is Landau damped ($\omega_i < 0$) or Landau growing ($\omega_i > 0$). For a Maxwellian, we have

$$\begin{aligned}
\omega_i &= \frac{\omega_{pe}^3\pi}{2k^2}\frac{1}{\sqrt{2\pi}v_{Te}}\frac{\partial}{\partial u}\left(\exp\left(-\frac{u^2}{2v_{Te}^2}\right)\right)\bigg|_{u=\omega_r/k} && (2.288)\\
&= -\frac{\omega_{pe}^3}{k^2}\sqrt{\frac{\pi}{8}}\frac{1}{v_{Te}^3}\frac{\omega_r}{k}\exp\left(-\frac{\omega_r^3}{2k^2v_{Te}^2}\right) && (2.289)\\
&= -\sqrt{\frac{\pi}{8}}\frac{\omega_r}{\lambda_D^3k^3}\exp\left(-\frac{\omega_{pe}^2+3k^2v_{Te}^2}{2k^2v_{Te}^2}\right) && (2.290)\\
&= -\sqrt{\frac{\pi}{8}}\frac{\omega_r}{\lambda_D^3k^3}\exp\left(-\frac{1}{2k^2\lambda_D^2}\right)\exp(-3/2) && (2.291)\\
&\approx -\sqrt{\frac{\pi}{8}}\frac{\omega_{pe}}{\lambda_D^3k^3}\exp\left(-\frac{1}{2k^2\lambda_D^2}\right)\exp(-3/2). && (2.292)
\end{aligned}$$

Therefore, for the Maxwellian case $\omega_i < 0$ and we have damping. The damping vanishes for $k \to 0$ and it increases with increasing k. For $k\lambda_D \sim 1$, i.e. short wavelengths, the damping is $\sim \omega_{pe}$ which is very large. Hence, such waves are never observed.

The complex angular frequency (2.287) can be understood on a more intuitive level. Any distribution function $g(u)$ that has $\partial g/\partial u|_{u=\omega_r/k} < 0$, i.e. a negative slope at the phase velocity ω_r/k of the wave, has more particles with slightly lower velocity than the phase velocity than it has particles with slightly higher velocity. It is mainly these particles, with velocity around the phase velocity of the wave, that interact with the wave. Particles with lower velocity get sped up by gaining energy from the wave. Particles with higher velocity get slowed down by giving up energy to the wave. Since there are more slower particles (due to negative slope) more particles are accelerated than decelerated meaning there is a net transport of energy from the wave to the particles and the wave is damped. This chain of thought can be repeated for distribution functions with positive slopes, in which case energy transitions to the wave causing the wave to grow.

2.4 From kinetic theory to fluid equations

Kinetic theory describes plasma in terms of the phase space distribution function. However, what's observed in experiments are quantities like particle density, temperature, rotation velocity, etc. In principle, we could always start with kinetic theory and derive these quantities. However, although neglecting velocity space effects, it is often sufficient to describe the plasma dynamics by means of the macroscopic quantities.

The fluid description frequently assumes that the velocity distribution of particles is given by a Maxwellian, that is, it is assumed that the velocity distribution is in equilibrium. This allows to define the state of the velocity distribution uniquely by one number, the temperature. But, since collisions can be rare in high temperature plasmas, deviations from the thermal equilibrium may remain for long periods of time. Surprisingly, the fluid description provides in many scenarios a robust and experimentally confirmed description.

Note that fluid theory comes also in multiple complexities. The fluid picture sees the plasma as a conglomerate of multiple interacting fluids. For example, a single ion-electron plasma can be described by two fluids. Most prominently, **magnetohydrodynamics** (MHD) combines all species of a plasma into a single fluid. This can be done including **resistive** effects or neglecting them (**ideal**). In the following, we will derive the two-fluid equations, also called Braginskii equations, from the Vlasov equation.

We transition from the phase-space Vlasov description of the plasma to the configuration space fluid picture by taking velocity moments of the distribution function and the Vlasov equation. By taking the moments, we get an infinite hierarchy of equations in 4D space since in every equation for a given moment the next order moment occurs. However, with an appropriate truncation, we will arrive at a suitable set of equations for two-fluid theory.

We start by noting that

$$n_\sigma(\mathbf{x}, t) = \int \mathrm{d}v^3 \; f_\sigma(\mathbf{x}, \mathbf{v}, t) \tag{2.293}$$

constitutes a normalization factor which at the same time is the particle density. Further, we define the fluid velocity

$$\mathbf{u}_\sigma(\mathbf{x}, t) = \frac{1}{n_\sigma} \int \mathrm{d}^3v \; \mathbf{v} f_\sigma(\mathbf{x}, \mathbf{v}, t). \tag{2.294}$$

With this definitions let's build the first moment of the Vlasov equation. Hence, we formally multiply the equation by 1 (first moment) and integrate over velocity

$$\int \mathrm{d}v^3 \left(\underset{①}{\partial_t f_\sigma} + \underset{②}{\mathbf{v} \cdot \nabla f_\sigma} + \frac{Z_\sigma e}{m_\sigma} \left(\underset{③}{\mathbf{E}} + \underset{④}{\frac{1}{c} \mathbf{v} \times \mathbf{B}} \right) \cdot \nabla_\mathbf{v} f_\sigma \right) = 0. \tag{2.295}$$

Integrating over the first term gives just $\partial_t n_\sigma$. For the second term, we have

$$\begin{aligned}
\textcircled{2} &= \int \mathrm{d}v^3\ \mathbf{v}\cdot\nabla f_\sigma && (2.296)\\
&= \int \mathrm{d}v^3\ \nabla\cdot(\mathbf{v}f_\sigma) && (2.297)\\
&= \nabla\cdot\int \mathrm{d}v^3\ \mathbf{v}f_\sigma && (2.298)\\
&= \nabla\cdot(n_\sigma\mathbf{u}_\sigma). && (2.299)
\end{aligned}$$

The third term gives

$$\begin{aligned}
\textcircled{3} &= \int \mathrm{d}v^3\ \mathbf{E}\cdot\nabla_{\mathbf{v}} f_\sigma && (2.300)\\
&= \int_\Omega \mathrm{d}v^3\ \nabla_{\mathbf{v}}\cdot(\mathbf{E}f_\sigma) && (2.301)\\
&\overset{\text{Gauss law}}{=} \int_{\partial\Omega} \mathrm{d}\mathbf{v}\cdot\mathbf{E}f_\sigma && (2.302)\\
&= \mathbf{E}\cdot\int_{\partial\Omega} \mathrm{d}\mathbf{v}\ f_\sigma && (2.303)\\
&= 0 && (2.304)
\end{aligned}$$

since the distribution function goes to zero for $\mathbf{v}\to\infty$. Since $\mathbf{v}\times\mathbf{B}$ commutes with $\nabla_{\mathbf{v}}$ the same holds true for the $\textcircled{4}$ term. Hence, we arrive at the

Continuity equation

$$\frac{\partial n_\sigma}{\partial t} + \nabla\cdot(n_\sigma\mathbf{u}_\sigma) = 0. \qquad (2.305)$$

This describes the conservation of particles. Also, we see that this equation contains the fluid velocity $\mathbf{u}_\sigma$. We will see in a moment that the next order equation provides the dynamical description of this quantity. The next higher moment of the Vlasov equation is determined by mutliplying the equation with $\mathbf{v}$ and integrating over velocity,

$$\int \mathrm{d}v^3\ \mathbf{v}\left(\underset{\textcircled{1}}{\partial_t f_\sigma} + \underset{\textcircled{2}}{\mathbf{v}\cdot\nabla f_\sigma} + \frac{Z_\sigma e}{m_\sigma}\left(\underset{\textcircled{3}}{\mathbf{E}} + \underset{\textcircled{4}}{\frac{1}{c}\mathbf{v}\times\mathbf{B}}\right)\cdot\nabla_{\mathbf{v}} f_\sigma\right) = 0. \qquad (2.306)$$

In this case, we have

$$\begin{aligned}
\textcircled{1} &= \int \mathrm{d}^3v\ \mathbf{v}\partial f_\sigma && (2.307)\\
&= \partial_t\int \mathrm{d}^3v\ \mathbf{v}f_\sigma && (2.308)\\
&= \partial_t(n_\sigma\mathbf{u}_\sigma). && (2.309)
\end{aligned}$$

For the next integral, we introduce a coordinate transformation in the integration, viz. $\mathbf{v} = \mathbf{v}'(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t)$. Herein, $\mathbf{u}_\sigma$ is the average part of the flow, while $\mathbf{v}'$ are random fluctuations. Note that $\mathbf{v}'$ depends on $\mathbf{x}$ while $\mathbf{v}$ doesn't. Also, it holds that $\mathrm{d}^3v = \mathrm{d}^3v'$. With this, we have

$$\begin{aligned}
â‘¡ &= \int \mathrm{d}^3v\ \mathbf{v}(\mathbf{v}\cdot\nabla) f_\sigma && (2.310)\\
&= \nabla \cdot \int \mathrm{d}^3v\ \mathbf{v}\mathbf{v} f_\sigma && (2.311)\\
&= \nabla \cdot \int \mathrm{d}^3v'\ \left(\mathbf{v}'\mathbf{v}' + \mathbf{v}'\mathbf{u} + \mathbf{u}\mathbf{v}' + \mathbf{u}\mathbf{u}\right) f_\sigma && (2.312)\\
&= \frac{1}{m_\sigma}\nabla\cdot\overleftrightarrow{\mathbf{P}} + \nabla\cdot(\mathbf{u}\mathbf{u}n_\sigma), && (2.313)
\end{aligned}$$

where we defined the **pressure tensor**

$$\overleftrightarrow{\mathbf{P}}_\sigma = m_\sigma \int \mathrm{d}^3v'\ \mathbf{v}'\mathbf{v}' f_\sigma. \tag{2.314}$$

The pressure is given by fluctuations in the velocity, while average flows do not contribute to the pressure. Also, we used here that $\int \mathrm{d}v'\mathbf{v}' f_\sigma = 0$ which is valid for symmetric distribution functions, which is the case for the Maxwellian. Further, we have

$$\begin{aligned}
â‘¢ &= \int \mathrm{d}^3v\ \mathbf{v}\mathbf{E}\cdot\nabla_\mathbf{v} f_\sigma && (2.315)\\
&= \int \mathrm{d}^3v\ \mathbf{v}\nabla_\mathbf{v}\cdot(\mathbf{E}f_\sigma) && (2.316)\\
&\overset{\text{P.I.}}{=} -\int \mathrm{d}^3v\ \mathbf{E}f_\sigma = -\mathbf{E}n_\sigma, && (2.317)
\end{aligned}$$

and

$$\begin{aligned}
â‘£ &= \int \mathrm{d}^3v\ \mathbf{v}(\mathbf{v}\times\mathbf{B})\cdot\nabla_\mathbf{v} f_\sigma && (2.318)\\
&= \int \mathrm{d}^3\ \mathbf{v}\nabla_\mathbf{v}\cdot\Big((\mathbf{v}\times\mathbf{B})f_\sigma\Big) && (2.319)\\
&\overset{\text{P.I.}}{=} -\int \mathrm{d}^3v\ \mathbf{v}\times\mathbf{B}f_\sigma && (2.320)\\
&= -\mathbf{u}\times\mathbf{B}n_\sigma. && (2.321)
\end{aligned}$$

Putting everything together, we arrive at the equation for momentum conservation

$$\partial_t(n_\sigma\mathbf{u}_\sigma) + \nabla\cdot(n_\sigma\mathbf{u}\mathbf{u}) = -\frac{1}{m_\sigma}\nabla\cdot\overleftrightarrow{\mathbf{P}}_\sigma + \frac{Z_\sigma e}{m_\sigma}n_\sigma\left(\mathbf{E} + \frac{1}{c}\mathbf{u}_\sigma\times\mathbf{B}\right). \tag{2.322}$$

By multiplying this equation with the mass m_σ, we see that it has units of *force/volume*. Hence, this equation is often also called the force balance equation.

Note that we didn't take collisions into account. Including a collision operator would introduce an additional term in the force balance equation, namely, a friction force term. That is, collisions between particles result in frictional flow of the fluid.

We can further develop the left hand side by

$$\partial_t(n_\sigma \mathbf{u}_\sigma) + \nabla\cdot(n_\sigma \mathbf{u}\mathbf{u}) = \underline{\mathbf{u}_\sigma \partial_t n_\sigma} + n_\sigma \partial_t \mathbf{u}_\sigma + n_\sigma(\mathbf{u}_\sigma\cdot\nabla)\mathbf{u}_\sigma + \underline{(\mathbf{u}_\sigma\cdot\nabla)(n_\sigma \mathbf{u}_\sigma)}, \tag{2.323}$$

where we see that the underlined terms are just (2.305)$\cdot\mathbf{u}_\sigma$ and thus cancel. The remainder is then

$$n_\sigma \partial_t \mathbf{u}_\sigma + n_\sigma(\mathbf{u}_\sigma\cdot\nabla)\mathbf{u}_\sigma = n_\sigma(\partial_t \mathbf{u}_\sigma + \mathbf{u}_\sigma\cdot\nabla\mathbf{u}_\sigma) \tag{2.324}$$

$$= n_\sigma \frac{\mathrm{d}\mathbf{u}_\sigma}{\mathrm{d}t}, \tag{2.325}$$

where we defined the **convective derivative**

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \mathbf{u}_\sigma\cdot\nabla. \tag{2.326}$$

Plugging this result into equation (2.322) and multiplying the equation with the mass, we arrive at the

Force balance equation

$$m_\sigma n_\sigma \frac{\mathrm{d}\mathbf{u}_\sigma}{\mathrm{d}t} = -\nabla\cdot\overleftrightarrow{\mathbf{P}}_\sigma + Z_\sigma e n_\sigma\left(\mathbf{E} + \frac{1}{c}\mathbf{u}_\sigma\times\mathbf{B}\right). \tag{2.327}$$

If f_σ is isotropic in the velocity which is the case for a Maxwellian distribution, the off-diagonal elements of the pressure tensor vanish [10]. Hence, it is useful to define the **scalar pressure**

$$p_\sigma = m_\sigma \int \mathrm{d}^3v'\, v'_x v'_x f_\sigma = m_\sigma \int \mathrm{d}^3v'\, v'_y v'_y f_\sigma = m_\sigma \int \mathrm{d}^3v'\, v'_z v'_z f_\sigma \tag{2.328}$$

$$= \frac{m_\sigma}{3}\int \mathrm{d}^3v'\, \mathbf{v}'\cdot\mathbf{v}' f_\sigma. \tag{2.329}$$

Usually, the scalar pressure is taken as the ideal gas pressure

$$p_\sigma = n_\sigma T_\sigma. \tag{2.330}$$

However, strictly speaking, it is only valid in a collisional plasma in thermodynamic equilibrium (following a Maxwellian) when thermal motion dominates over collective effects and the magnetic field does not introduce significant anisotropies.

[10] Note that in the plasma wave section we assumed that the plasma is perturbed from a Maxwellian equilibrium. Hence, the perturbation δf to the distribution function may introduce an anisotropic contribution to the pressure.

With the scalar pressure, we can dissect the pressure tensor in two parts

$$\overleftrightarrow{\mathbf{P}}_\sigma = p_\sigma \mathbb{1} + \overleftrightarrow{\mathbf{\Pi}}, \tag{2.331}$$

where $\overleftrightarrow{\mathbf{\Pi}}$ is the off-diagonal, or anisotropic part of the pressure tensor and is called the **viscous stress tensor**. The viscous stress tensor is usually smaller than the scalar pressure and you will find it often omitted from the force balance equation. As the name suggests, this anisotropic part introduces viscosity.

As mentioned before, the equations we get by performing moments of the Vlasov equation always depend on moments that are one order higher. For example, in the continuity equation (2.322), the fluid velocity $\mathbf{u} \propto \int \mathrm{d}v \mathbf{v} f$ occurs, which is already the first velocity moment. For the force balance equation, it is the pressure tensor which is one order higher. Hence, to find an equation for the pressure tensor, the next order moment equation has to be determined. This next order equation describes the conservation of energy and gives the dynamical description of the temperature.

This hierarchy of equations never leads to an end, thus an exact description requires an infinite number of equations. However, we can stop at any point by making an ad hoc closing assumption.

Of course, self-consistency of the fluid plasma description requires Maxwell's equations. In the fluid case, the charge and current densities are given by

$$\rho_\sigma(\mathbf{x}, t) = Z_\sigma e \int \mathrm{d}^3 v \; f_\sigma(\mathbf{x}, \mathbf{v}, t) \tag{2.332}$$

$$= Z_\sigma e n_\sigma(\mathbf{x}, t), \tag{2.333}$$

$$\mathbf{j}_\sigma(\mathbf{x}, t) = Z_\sigma e \int \mathrm{d}^3 v \; \mathbf{v} f_\sigma(\mathbf{x}, \mathbf{v}, t) \tag{2.334}$$

$$= Z_\sigma e \mathbf{u}_\sigma(\mathbf{x}, t). \tag{2.335}$$

Note that in this derivation, we omitted collisions. In this case, different fluids only couple via the charge and current densities and the electromagnetic fields. However, collisions would introduce an additional force term in the force balance equation that corresponds to **collisional friction**. This collisional friction on the other hand also introduces electrical resistivity. Hence, without collisions considered, the fluid equations derived here are "ideal", i.e. zero resistivity or infinite conductivity.

After deriving the fluid equations, where we have a set of equations for each individual particle species, we can go a step further in the simplification and describe the whole system as a single fluid. The resulting theory is called **magnetohydrodynamics** (MHD). The advantage of this description is its simplicity, but it only describes large-scale, low-frequency phenomena. As such, it is frequently used to determine magnetic equilibrium configurations in magnetic confinement devices. In particular, we define the total mass density

$$\rho = \sum_\sigma m_\sigma n_\sigma, \tag{2.336}$$

the total current density

$$\mathbf{J} = \sum_\sigma Z_\sigma e n_\sigma \mathbf{u}_\sigma, \tag{2.337}$$

and the center of mass fluid velocity

$$\mathbf{V} = \frac{1}{\rho} \sum_\sigma m_\sigma n_\sigma \mathbf{u}_\sigma. \tag{2.338}$$

If we sum up the force balance equation for each species, for example, we get the force balance equation of MHD

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla\right)\mathbf{V} = -\nabla p + \frac{1}{c}\mathbf{J} \times \mathbf{B}. \tag{2.339}$$

The equilibrium of a fusion device is then determined by the stationary steady-state of this equation, that is

$$c\nabla p = \mathbf{J} \times \mathbf{B}. \tag{2.340}$$

This equation looks deceptively simple, but it can be rather involved to solve for realistic configurations.

2.5 Collisions

Collisions in a fully-ionized plasma are given by **Coulomb interactions** between charged particles [11]. The effect of collisions is the transfer of energy and momentum between particles. Most importantly, collisions lead to a redistribution of the particle distribution function towards **thermal equilibrium**, i.e. a Maxwellian distribution. Collisions, and thermal equilibration, happens **within species**, e.g. electrons and electrons, and also **between species**, e.g. electrons and ions. The latter is important for the thermalization of the plasma as a whole.

The frequency of collisions depends on the parameters of a plasma, in particular, density and temperature. A higher density provides more particles that can interact and collide with each other. On the other hand, higher temperature means a larger average velocity (v_T) and, thus, particles spend less time in the Coulomb "influence" zone of other particles given by the Debye length. Without deriving this here, the scaling of the (perpendicular momentum) collision frequency is given by

Collision frequency scaling

$$\nu \propto \frac{n}{T^{3/2}}. \tag{2.341}$$

[11] Head on hard-sphere collisions like with neutral particles are only occurring in partially-ionized plasmas where there are still neutral particles present.

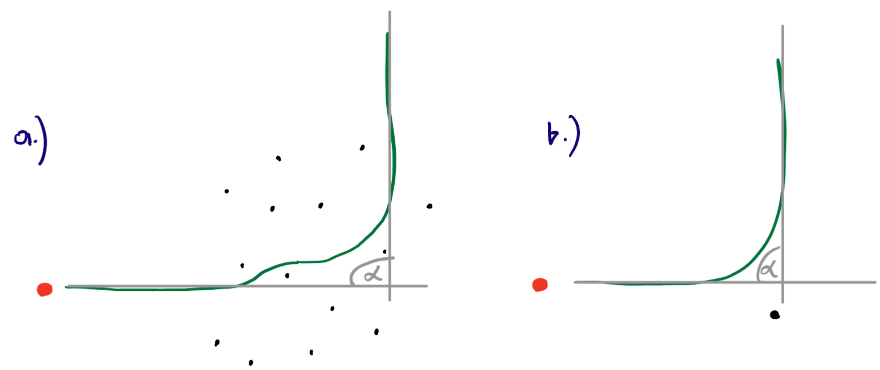

Figure 2.10: Sketch of the two types of collisions: a.) multiple small-angle scatterings, b.) single large-angle scattering.

There are in fact multiple definitions of the collision frequency, or even what constitutes a "collision". Most often, a Coulomb interaction between particles is considered a collision in the proper sense if the scattering angle, i.e. the angle between the momentum before and after the interaction, is larger than 90°. There are two options to arrive at this which are sketched in figure 2.10: either by a single large-angle scattering or by many small-angle scatterings. The latter is the more frequent case.

The collision frequency will occur in the kinetic equation as part of the collision operator on the right hand side. As mentioned before, the right hand side of the kinetic equation describes the particle-like behavior of the plasma, i.e. the collisions between particles. The left hand side describes the wave-like behavior of the plasma, i.e. the collective motion of particles in response to electromagnetic fields.

Including collisions in Vlasov's equation results in the plasma kinetic equation,

Plasma kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m}\Big(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\Big) \cdot \nabla_{\mathbf{v}} f = \Big(\frac{\partial f}{\partial t}\Big)_{\text{coll.}}, \tag{2.342}$$

where we now have a source term on the right hand side representing collisions. This term is often written as an operator

$$\Big(\frac{\partial f}{\partial t}\Big)_{\text{coll.}} = C(f), \tag{2.343}$$

acting on the particle distribution function. Particles can collide with other particles of the same species, or with particles from other species. Hence, we can write the collision operator as a sum

$$C(f_\alpha) = \sum_\beta C(f_\alpha, f_\beta), \tag{2.344}$$

indicating that species alpha collides with all other species.

Conservation rules

The collision operator should fulfill certain conservation rules [5] motivated by physics. For example, collisions only change the velocity of the colliding particles, not their position, i.e. they don't get "teleported". Therefore, the particle density should not be affected by collisions [12]. Hence, considering the fluid continuity equation including the collision term, we have

$$\frac{\partial n_\sigma}{\partial t} + \nabla \cdot (n_\sigma \mathbf{u}_\sigma) = \left.\frac{\partial n_\alpha}{\partial t}\right|_{\text{coll.}} = \int \mathrm{d}^3v \, \sum_\beta C(f_\alpha, f_\beta) = 0. \tag{2.345}$$

Of course, strictly speaking, we could include the impact of fusion processes, in which case particles of one species are destroyed while particles of another species are created. However, fusion processes even in planned fusion reactors are usually so rare that this effect can be neglected.

Further, the collision operator has to follow conservation laws for momentum and energy,

$$\int \mathrm{d}^3v \; m_\alpha \mathbf{v} C(f_\alpha, f_\beta) = -\int \mathrm{d}^3v \; m_\beta \mathbf{v} C(f_\beta, f_\alpha), \tag{2.346}$$

$$\int \mathrm{d}^3v \; \frac{m_\alpha v^2}{2} C(f_\alpha, f_\beta) = -\int \mathrm{d}^3v \; \frac{m_\beta v^2}{2} C(f_\beta, f_\alpha). \tag{2.347}$$

This implies that the force species α exerts on β is equal and opposite to that which β exerts on α, and that no energy is produced by collisions (again neglecting fusion processes). Intra-species collisions ($\alpha = \beta$) gives

$$\int \mathrm{d}^3v \; m_\alpha \mathbf{v} C(f_\alpha, f_\alpha) = 0, \tag{2.348}$$

$$\int \mathrm{d}^3v \; \frac{m_\alpha v^2}{2} C(f_\alpha, f_\alpha) = 0. \tag{2.349}$$

The most important effect of the collision operator is that it drives the distribution function towards local thermodynamic equilibrium, that is, towards a Maxwellian. Hence, any distribution of particles in a plasma, without driving mechanisms, will eventually end up in thermodynamic equilibrium.

[12] As a sudden change in position would "teleport" a particle to a different position, which could be well outside the plasma and thus changing the density.

2.5.1 Bhatnagar–Gross–Krook collision operator

Of course, we need a mathematical description of the collision operator. Various operators exist in the literature. Probably the most simplest one is the Bhatnagar–Gross–Krook (BGK) collision operator. It is given by

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll.}} = -\nu\Big(f(\mathbf{x},\mathbf{v},t) - f_0(\mathbf{v})\Big). \tag{2.350}$$

This operator does not conserve energy nor momentum, but only the particle number. Still, due to its simplicity it is used to incorporate thermalization of the plasma. The BGK operator drives the distribution function towards a Maxwellian f_0 with a collision frequency ν. The BGK operator is a linear operator, i.e. it can be applied to any distribution function and will always drive it towards the Maxwellian.

2.5.2 Fokker-Planck collision operator

An equation describing the temporal evolution of the particle distribution function due to the cumulative effect of many collisions was found by Adriaan Fokker [2] and Max Planck [9] (though many more names for this equation in different contexts are known, see wikipedia). The Fokker-Planck equation is not specific to plasma physics. Here, we use the Fokker-Planck equation to describe the impact of cumulative small angle scattering by Coulomb collisions on f, that is, we find an expression for $(\partial_t f)_{coll.}$. We will see, that the resulting equation describes the time evolution of f in terms of **drag** and **diffusion**. Before specifying the discussion to plasmas, let's derive the Fokker-Planck equation generally [11].

General derivation

We start by writing

$$\left(\frac{\partial f}{\partial t}\right)_{coll.} = \frac{f(\mathbf{x},\mathbf{v},t+\Delta t) - f(\mathbf{x},\mathbf{v},t)}{\Delta t}. \tag{2.351}$$

The change in the distribution function at the later time step can be found by integrating over all possible changes in the velocity

$$f(\mathbf{x},\mathbf{v},t+\Delta t) = \int \mathrm{d}\Delta\mathbf{v}\; f(\mathbf{x},\mathbf{v}-\Delta\mathbf{v},t)\Psi(\mathbf{v}-\Delta\mathbf{v},t), \tag{2.352}$$

where $\Psi(\mathbf{v}-\Delta\mathbf{v},\Delta\mathbf{v})$ is the probability that a particle with a velocity $\mathbf{v}$ changes its velocity by $\Delta\mathbf{v}$ due to collisions in a time Δt. Note that the probability has to sum up to one, i.e. $\int \mathrm{d}\Delta\mathbf{v}\Psi(\mathbf{v},\Delta\mathbf{v}) = 1$. Since most collisions are small angle scatterings, and thus change the velocity only little[13], we can Taylor expand the

[13]Actually, the derivation does not require this assumption, but we make it anyway in respect to the application to plasmas.

integrand. Thus,

$$f(\mathbf{x}, \mathbf{v} - \Delta\mathbf{v}, t)\Psi(\mathbf{v} - \Delta\mathbf{v}, t) = f(\mathbf{x}, \mathbf{v}, t)\Psi(\mathbf{v}, \Delta\mathbf{v}) - \sum_i \frac{\partial}{\partial v_i}(f\Psi)\Delta v_i + \frac{1}{2}\sum_{ij}\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}(f\Psi)\Delta v_i \Delta v_j. \tag{2.353}$$

Plugging this into the integral, we have

$$\begin{aligned} f(\mathbf{x}, \mathbf{v}, t + \Delta t) &= f(\mathbf{x}, \mathbf{v}, t) - \sum_i \frac{\partial}{\partial v_i} f(\mathbf{x}, \mathbf{v}, t) \int \mathrm{d}\Delta\mathbf{v}\ \Psi(\mathbf{v}, \Delta\mathbf{v})\Delta v_i \\ &+ \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial v_i \partial v_j} f(\mathbf{x}, \mathbf{v}, t) \int \mathrm{d}\Delta\mathbf{v}\Psi(\mathbf{v}, \Delta\mathbf{v})\Delta v_i \Delta v_j \\ &= f(\mathbf{x}, \mathbf{v}, t) - \sum_i \frac{\partial}{\partial v_i} f(\mathbf{x}, \mathbf{v}, t)\langle \Delta v_i \rangle \\ &+ \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial v_i \partial v_j} f(\mathbf{x}, \mathbf{v}, t)\langle \Delta v_i \Delta v_j \rangle, \end{aligned} \tag{2.354}$$

where we defined the drag coefficient (dynamic friction)

$$\langle \Delta v_i \rangle = \int \mathrm{d}\Delta\mathbf{v}\ \Psi(\mathbf{v}, \Delta\mathbf{v})\Delta v_i, \tag{2.355}$$

and the diffusion tensor

$$\langle \Delta v_i \Delta v_j \rangle = \int \mathrm{d}\Delta\mathbf{v}\ \Psi(\mathbf{v}, \Delta\mathbf{v})\Delta v_i \Delta v_j. \tag{2.356}$$

Finally, we arrive at the Fokker-Planck equation

Fokker-Planck equation

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{c.}} = -\sum_i \frac{\partial}{\partial v_i}\Big(f(\mathbf{x}, \mathbf{v}, t)A_i(\mathbf{v})\Big) + \sum_{i,j}\frac{\partial^2}{\partial v_i \partial v_j}\Big(f(\mathbf{x}, \mathbf{v}, t)D_{ij}(\mathbf{v})\Big), \tag{2.357}$$

where

$$A_i(\mathbf{v}) = \frac{\langle \Delta v_i \rangle}{\Delta t}, \tag{2.358}$$

$$D_{ij}(\mathbf{v}) = \frac{\langle \Delta v_i \Delta v_j \rangle}{\Delta t}. \tag{2.359}$$

The first term in this equation describes the change of f due to the average change in v [5], i.e. the average force a particle feels due to collisions. It has the effect of a **drag** on the particle and thus slows it down. This force pulls toward $\mathbf{v} = 0$. This pull is balanced by the second term that describes random and **diffusive** spreading in velocity space. The diffusion coefficient is given by (assuming only one spatial dimension for a moment)

$$D = \frac{\langle \Delta v^2 \rangle}{2\Delta t} \sim \frac{\text{step size}^2}{\text{time step}}. \tag{2.360}$$

Note that the diffusion coefficient is a tensor indicating a possibly non-isotropic behavior. The balance of drag force and diffusive spreading results in a Maxwellian distribution (2.72) in equilibrium.

In general, the coefficients A_i and D_{ij} incorporate the effects of Coulomb collisions.

As is apparent, the combining the Fokker-Planck collision operator with the Vlasov equation results in a highly complex partial differential equation. Consequently, various approximations and simplifications have been developed to make the Fokker-Planck equation more tractable. An example is given by the Ornstein-Uhlenbeck approximation, which assumes a linear drag force and constant diffusion coefficient. This leads to a simplified form of the Fokker-Planck equation that is easier to analyze and solve.

Ornstein-Uhlenbeck approximation

Assuming the drag force coefficient to be linear in v and the diffusion to be constant, we can write down the Ornstein-Uhlenbeck form of the Fokker-Planck equation. In 1D it is given by

$$\left(\frac{\partial f}{\partial t}\right)_{\text{c.}} = \nu \frac{\partial}{\partial v}\left(vf + v_T^2 \frac{\partial}{\partial v} f\right) \tag{2.361}$$

where we expressed the coefficients by

$$A = -v\nu \tag{2.362}$$

$$B = v_T^2 \nu. \tag{2.363}$$

2.6 Questions Kinetic Theory

Vlasov equation

(2.1) Derive the Klimontovich equation from the multi-particle phase space density.

(2.2) What is the issue with the Klimontovich equation?

(2.3) Derive the Boltzmann-Vlasov equation from the Klimontovich equation.

(2.4) What does the Boltzmann equation describe?

(2.5) How does the Vlasov equation relate to the Boltzmann equation?

(2.6) What is a general property of equilibrium solutions to the Vlasov equation? How does the equilibrium solution relate to the motion of individual particles?

(2.7) What does the Maxwell-Boltzmann distribution describe? Sketch and discuss its properties.

(2.8) Tell me everything you know about the plasma kinetic equation (e.g. properties of equilibrium solutions, how to get plasma waves, how it is related to the particle and fluid pictures, ...).

Plasma waves

Linear plasma waves in general

(2.9) What are the assumptions to describe linear plasma waves?

(2.10) What impact does the plasma have on electromagnetic waves?

(2.11) How are the plasma conductivity tensor, the dielectric tensor and the refraction index related?

(2.12) Discuss the dispersion equation (derivation, result, etc.).

(2.13) What effect does the refraction index have on the wave? Discuss limiting cases.

(2.14) What does the dispersion relation tell you? What types are there (e.g. dispersive, linear, etc.)?

Cold plasma waves

(2.15) What are the assumptions of cold plasma waves?

(2.16) Sketch the derivation of the dispersion equation. How does the cold plasma assumption impact the derivation?

(2.17) Discuss the dielectric tensor (e.g. anisotropy, gyrotropic, etc.).

(2.18) How can we tell which types of plasma waves exist? How do we know their allowed frequency range and polarization?

(2.19) What are cut-offs and resonances?

(2.20) Tell me some types of plasma waves for parallel propagation and discuss them.

(2.21) Tell me some types of plasma waves for perpendicular propagation and discuss them.

Warm and hot plasma waves

(2.22) Show the derivation of the electrostatic dispersion equation.

(2.23) Give the electrostatic dispersion equation and discuss the options you have to solve it.

(2.24) What are Langmuir waves?

(2.25) Sketch the steps to derive the dispersion relation for warm Langmuir waves.

(2.26) What is the dispersion relation of warm Langmuir waves? How does it differ to cold plasma waves?

(2.27) What are phase and group velocity of warm Langmuir waves?

(2.28) Describe Landau damping or growth. How can we understand it physically?

(2.29) What is the difference in the dispersion relation of Langmuir waves in the cold, warm and hot plasma limits? What are the individual assumptions?

Collisions

(2.30) How are collisions defined?

(2.31) What impact do collisions have on the plasma?

(2.32) What rules does the collision operator have to follow? Why are they important?

(2.33) Derive the Fokker-Planck equation. What are the individual terms of the final expression describing?

(2.34) Write down the Fokker-Planck equation. What are the terms describing?

(2.35) What is the equilibrium solution to the Fokker-Planck equation?

From kinetic to fluid equations

(2.36) How can we derive the fluid equations from the kinetic equation? What are the three most prominent resulting fluid equations?

(2.37) Show the derivation of the continuity equation from the Vlasov equation.

Chapter 3

Advanced topics

3.1 Drift-kinetic theory

3.1.1 Derivation of the drift-kinetic equation

An important approximation of the kinetic equation is the drift-kinetic equation. This form is particularly prominent in transport models[5]. To derive it, we start with the kinetic equation in the form

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{z}} \cdot (\dot{\mathbf{z}} f) = C(f). \tag{3.1}$$

Here, $\mathbf{z} = (\mathbf{x}, \mathbf{p})$ is the 6D phase space variable and $\nabla_{\mathbf{z}} = \partial/\partial\mathbf{z}$. Without collisions, this equation would be a continuity equation of the particle distribution function in phase space where $\hat{\mathbf{z}} f$ is the particle flux. With the help of Hamilton's equations it is straightforward to show that the phase space flow velocity is divergence free

$$\nabla_{\mathbf{z}} \dot{\mathbf{z}} = 0. \tag{3.2}$$

Therefore, the kinetic equation can be written as

$$\frac{\partial f}{\partial t} + \dot{z}_k \frac{\partial f}{\partial z_k} = C(f). \tag{3.3}$$

Since the scalar product ensures diffeomorphism invariance, we can change the coordinate system to the guiding-center variables $\mathbf{w} = (\mathbf{R}, \mathcal{E}, \mu, \vartheta)$. The momenta are now given by the energy

$$\mathcal{E} = \frac{mv^2}{2} + Ze\Phi, \tag{3.4}$$

the magnetic moment $\mu = mv_\perp^2/(2B)$ and the gyrophase ϑ. Using this set of variables, we have

$$\frac{\partial f}{\partial t} + \dot{\mathbf{R}} \cdot \nabla f + \dot{\mathcal{E}} \frac{\partial f}{\partial \mathcal{E}} + \dot{\mu} \frac{\partial f}{\partial \mu} + \dot{\vartheta} \frac{\partial f}{\partial \vartheta} = C(f). \tag{3.5}$$

In a fusion plasma, the magnetic moment μ is usually conserved and the corresponding term vanishes. Also, the last term on the left hand side is dominating over all other terms since $\dot{\vartheta} = \Omega_c$, where Ω_c is the cyclotron frequency which is large ($\approx 10^{10}$-10^{12} Hz). The guiding center velocity term, with $\dot{\mathbf{R}} = \mathbf{h}v_\parallel + \mathbf{v}_d$, is of the order d^{-1} where $\delta \equiv \max(\rho/L_B, 1/(\Omega_c\tau_B))$. Here, L_B and τ_B are the typical gradient scale and the typical temporal scale of the magnetic field. On the other hand, the collision term on the right hand side is proportional to Δ^{-1} with $\Delta = \nu/\Omega_c \ll 1$ being the ratio of the collision frequency and the gyrotron frequency, which is small in a fusion plasma.

As mentioned, drift-kinetic theory is particularly useful for studying slow transport phenomena. In this case, $\partial_t \sim \delta^2\nu$ and energy is conserved. In zeroth order of δ and Δ, we have

$$\frac{\partial f_0}{\partial \vartheta} = 0. \tag{3.6}$$

If we average the drift-kinetic equation over the gyrophase and assume slow dynamics, we have

$$\boxed{v_\parallel \nabla_\parallel f_0 + \mathbf{v}_d \cdot \nabla f_0 = C(f_0),} \tag{3.7}$$

which is the drift-kinetic equation used to study transport phenomena.

Note that the zeroth order distribution function $f_0 = f_0(\mathbf{R})$ rather than $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$ where $\boldsymbol{\rho} = \mathbf{h} \times \mathbf{v}/\Omega_c$ is the Larmor vector. Hence, we can write

$$f(\mathbf{r}) \simeq f_0(\mathbf{R}) \simeq f_0(\mathbf{r}) - \boldsymbol{\rho} \cdot \nabla f_0(\mathbf{r}), \tag{3.8}$$

which is a Taylor expansion in the Larmor radius. Note that if $f_0(\mathbf{R})$ is Maxwellian, the distribution function $f_0(\mathbf{r})$ will differ by $\mathcal{O}(\delta)$. In fact, the variation from the Maxwellian given by the term $-\boldsymbol{\rho} \cdot \nabla f_0$ is what causes diamagnetic fluxes of particles, momentum and heat. Furthermore, the relaxation to a Maxwellian is driven by classical cross-field transport. Since the Larmor radius is small, classical transport is weak in comparison to other transport mechanisms. Most importantly, neoclassical transport, introduced by non-straight magnetic field structures, is caused by guiding-center drifts and is much larger than classical transport.

Bounce-averaged drift-kinetic equation: If we are interested in even slower phenomena, we could do further averaging over the parallel particle motion. This would result in the bounce-averaged drift-kinetic equation, resulting in a 4-dim phase space. Not treated in this lecture.

3.1.2 Transport theory

Often, the subject of transport theory the radial fluxes of particles and heat. They can be brought into the form

$$\Gamma = -n(D_{11}A_1 + D_{12}A_2), \tag{3.9}$$

$$Q = -nT(D_{21}A_1 + D_{22}A_2). \tag{3.10}$$

where A_1 and A_2 are thermodynamic forces.

3.2 Gyro-kinetic theory

In drift-kinetic theory, it was assumed that the electromagnetic fields do not change much over one gyro period. However, there are phenomena that require loosing this assumption. In particular, plasma turbulence which is the dominating transport mechanism in magnetically confined fusion plasmas (even larger than neoclassical transport). To describe turbulence, which is a microscopic effect, the electromagnetic field variation over the gyromotion has to be considered. The result is gyro-kinetic theory.

3.2.1 Derivation of the gyro-kinetic equation

3.3 Bernstein waves

- Magnetized plasmas, i.e. includes gyro motion
- Cyclotron damping

3.3.1 Cyclotron damping

3.4 Hamiltonian form of the Vlasov equation

Another neat way to write the Vlasov equation is in terms of Hamiltonian mechanics. In particular, the time evolution of a quantity of a Hamiltonian system is given by the total time derivative. For the distribution function, this is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} - \{f, H\}, \tag{3.11}$$

which is either zero (no collisions) or equal to some collision term. Here, the bracket is the Poisson bracket defined by

$$\{f, H\} = \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{x}}, \tag{3.12}$$

where ($\mathbf{x}$, $\mathbf{p}$) are canonical coordinates. The Hamiltonain H appearing in the Poisson bracket is the single particle Hamiltonian corresponding to the macroscopic fields. Of course, f and H in equation (3.11) corresponds to one species σ.

3.4.1 Action-angle coordinates

Appendix A

Useful vector identities

- BAC-CAB rule

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{A.1}$$

Appendix B

Some basics on waves

Relaxation [10]: The particle distribution function tends towards an equilibrium solution (see section 2.2.1) due to collisions[1]. In an anisotropic plasma, e.g. in the presence of a magnetic field, the relaxation times in parallel and perpendicular direction may be different.

Thermalization [10]: Waves tend to thermalize, i.e. dissipate their energy to the particles. The timescale depends on the plasma parameters like density, temperature, etc. In a collisionless plasma, the timescale is longer since dissipation only occurs via wave-particle interaction (discussed in section 2.3.4). Thermalization is an important topic in the context of heating of fusion plasmas by electromagnetic waves (electron/ion cyclotron resonance heating).

Monochromatic waves: We consider here only monochromatic waves of the form

$$\mathbf{A}(\mathbf{x}, t) = \tilde{\mathbf{A}}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{B.1}$$

In this case, derivative operators take an algebraic form,

$$\frac{\partial}{\partial t} = \mathrm{i}\omega, \tag{B.2}$$

$$\nabla = -\mathrm{i}\mathbf{k}, \tag{B.3}$$

where $\mathbf{k}$ is the wave vector indicating the direction of propagation. When the wave number $k = |\mathbf{k}|$ is real, the wave propagates unattenuated with the phase velocity

$$v_p = \frac{\omega}{k}, \tag{B.4}$$

and the group velocity

$$v_g = \frac{\partial \omega}{\partial k}. \tag{B.5}$$

[1] TODO: Also, without collisions, the distribution function tends towards an equilibrium due to Boltzmann's H-theorem (entropy considerations).

Further, the wave length is defined by $\lambda = 2\pi/k$. If, however, the wave number is complex, $k = a - \mathrm{i}b$, the wave propagates with the phase velocity $v_p = a/k$ and the amplitude decreases continuously with $\exp(-b\mathbf{x})$ $(b > 0)$. If k is purely imaginary, the wave is evanescent.

Furthermore, the frequency of the wave could also be complex, $\omega = \omega_r + \mathrm{i}\omega_i$. As for a purely real frequency, the real part describes a usual oscillating wave. However, the imaginary part leads to a temporally damped or growing wave, depending on the sign of the imaginary part.

For comparison, a **wave packet** can be written as

$$\mathbf{A}(\mathbf{x}, t) = \int \mathrm{d}\omega \mathrm{d}^3k \ \tilde{\mathbf{A}}(\mathbf{k}) \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)}. \tag{B.6}$$

Linear theory [10]: Often, we assume the wave amplitude to be small compared to an equilibrium, e.g.

$$\mathbf{E} = \mathbf{E}_0 + \delta\mathbf{E}, \tag{B.7}$$

where $|\mathbf{E}_0| \gg |\delta\mathbf{E}|$. Then, by linearizing the governing equations, i.e. by substituting the linear form (B.7) for various quantities and neglecting terms of $\mathcal{O}(\delta^2)$, we can simplify the problem. In particular, if we assume a wave ansatz for the perturbation, the initial differential equations are cast into an algebraic form.

But, wave-particle interactions are generally non-linear. Still, linear treatment has a bounded region of validity due to collisions. The reasoning is the following. Consider a travelling potential well. Particles can get trapped in this well and cease to exchange energy with the wave on average. In this case, damping of the wave by Landau damping or cyclotron damping ceases. However, if collisions are considered, the particles can be removed from the well before they 'thermalize'. In this case, linear theory still gives an accurate treatment. Hence, collisions enable linear theory.

Wave vs. Mode

Often in magnetic confinement fusion physics, there is talk about modes, for example **ballooning modes** or **tearing modes**. But what is the difference to plasma waves?

Waves: propagate, i.e. move through the plasma.

Modes: spatial structure that is either stationary, growing or oscillating.

The fundamental plasma waves discussed here are naturally occurring in plasma due to the collective behaviour. In other words, they are driven by electromagnetic forces. Modes, like ballooning modes, are driven by kinetic[a] properties like the pressure gradient, or current gradient.

[a]"Kinetic" in this context means that it originates from the particles and their movement.

In general, waves and modes require a driving (or restoring) force that is hampered by an inertial property. For example, Langmuir waves or plasma waves are oscillations of electrons around a stationary ion background. Imagine pulling the electrons away from the ion background and let go. In this case, the restoring force is the Coulomb force mediated by the electric field and the inertia is provided by the electron mass. Some examples of force/inertia combinations are shown in table B.1. In this lecture, we will only discuss waves.

Table B.1: Comparison of restoring forces and inertia for different plasma waves and modes. TODO: Fully clarify driving forces and inertia

Wave/Mode	**Restoring Force**	**Inertia**
Langmuir Waves	Coulomb force (electric field)	Electron mass
Ion Acoustic Waves	Electron pressure gradient	Ion mass
Alfvén Waves	Magnetic tension (Lorentz force)	Plasma mass density ρ
Whistler Waves	Magnetic field restoring force [2]	Electron gyromotion (cyclotron motion)
Ballooning Modes	Plasma pressure gradient	Magnetic field line tension in plasma core
Peeling Modes	Plasma pressure gradient	Magnetic field line tension at plasma edge
Kink Modes	Current density gradient	Magnetic tension
Tearing Modes	Current density gradient	

[2]Magnetic field restoring force is a broader term than only magnetic tension. It includes both magnetic tension and magnetic pressure effects (if applicable).

Appendix C

Derivation of the guiding-center Lagrangian

In the following, we derive the guiding-center Lagrangian where we assume a (weakly) inhomogeneous magnetic field that is static, hence, $\mathbf{B} = \mathbf{B}(\mathbf{r}) = B(\mathbf{r})\mathbf{h}(\mathbf{r})$. The weak inhomogeneous criterion is given by

$$\frac{\rho_L}{L_B} \ll 1, \tag{C.1}$$

where $L_B = |\nabla \ln B|^{-1}$ is the magnetic field length scale.

First, we transition from the particle coordinates $\mathbf{z}_P = (\mathbf{r}, \mathbf{v})$ in phase space, to guiding-center coordinates $\mathbf{z} = (\mathbf{R}, \phi, v_\parallel, v_\perp)$. The transformation is given by

$$\mathbf{r}(\mathbf{z}) = \mathbf{R} + \boldsymbol{\rho}(\mathbf{R}, \phi, v_\perp), \tag{C.2}$$

$$\mathbf{v}(\mathbf{z}) = v_\parallel \mathbf{h}(\mathbf{R}) + v_\perp \hat{\mathbf{n}}(\mathbf{R}, \phi), \tag{C.3}$$

where

$$\boldsymbol{\rho}(\mathbf{R}, \phi, v_\perp) = \frac{mcv_\perp}{eB(\mathbf{R})} \hat{\boldsymbol{\rho}}(\mathbf{R}, \phi) \tag{C.4}$$

$$= \rho_L(\mathbf{R}, v_\perp) \hat{\boldsymbol{\rho}}(\mathbf{R}, \phi). \tag{C.5}$$

It is straightforward to show

$$\hat{\boldsymbol{\rho}}(\mathbf{R}, \phi) = -\frac{\partial \hat{\mathbf{n}}(\mathbf{R}, \phi)}{\partial \phi}, \tag{C.6}$$

$$\hat{\mathbf{n}}(\mathbf{R}, \phi) = \frac{\partial \hat{\boldsymbol{\rho}}(\mathbf{R}, \phi)}{\partial \phi}. \tag{C.7}$$

If the magnetic field is homogeneous, $(\phi, v_\parallel, v_\perp)$ are the gyro phase, the parallel and the perpendicular velocity of the particle. In the case of a weakly inhomogeneous field, they still are good approximations for the particle values averaged over one gyro-period. The important difference comes from the values which are different if taken at the particle position or the guiding-center position $\mathbf{R}$. Note that the latter coincides with the center of the gyration only in the homogeneous case. In the weakly inhomogeneous case, it is close and thus a good approximation.

Let us write the phase-space Lagrangian (see section D.1)

$$L(\mathbf{r}, \dot{\mathbf{r}}, \mathbf{v}, \dot{\mathbf{v}}) = \Big(m\mathbf{v} + \frac{e}{c}\mathbf{A}(\mathbf{r})\Big) \cdot \mathbf{r} - \Big(\frac{m}{2}v^2 + e\Phi(\mathbf{r})\Big) \tag{C.8}$$

in terms of the new coordinates

$$\begin{aligned} L(\mathbf{R}, \dot{\mathbf{R}}, \phi, \dot{\phi}, v_\parallel, v_\perp) = {} & \Big(mv_\parallel \mathbf{h}(\mathbf{R}) + mv_\perp \hat{\mathbf{n}}(\mathbf{R}, \phi) + \frac{e}{c}\mathbf{A}(\mathbf{R} + \boldsymbol{\rho})\Big) \cdot (\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}) \\ & - \Big(\frac{m}{2}(v_\parallel^2 + v_\perp^2) + e\Phi(\mathbf{R} + \boldsymbol{\rho})\Big). \end{aligned} \tag{C.9}$$

So far, this is an exact Lagrangian, just written in a different system of coordinates. Now, we introduce an approximation by Taylor expansion in the small parameter of

$$\rho_L = |\boldsymbol{\rho}| = \frac{mcv_\perp}{eB(\mathbf{R})} \sim \varepsilon, \tag{C.10}$$

where we introduce now an ordering parameter ε which we eventually set to unity. The gyro motion is not only small in spatial dimensions, but also fast in temporal. Hence, it introduces a large parameter compared to frequency scales,

$$\omega_c = \frac{eB(\mathbf{R})}{mc} \sim \varepsilon^{-1}. \tag{C.11}$$

The potentials are now expanded as

$$\mathbf{A}(\mathbf{R} + \boldsymbol{\rho}) = \mathbf{A}(\mathbf{R}) + \varepsilon\boldsymbol{\rho} \cdot \nabla\mathbf{A}(\mathbf{R}) + \mathcal{O}(\varepsilon^2) \tag{C.12}$$

$$\Phi(\mathbf{R} + \boldsymbol{\rho}) = \Phi(\mathbf{R}) + \varepsilon\boldsymbol{\rho} \cdot \nabla\Phi(\mathbf{R}) + \mathcal{O}(\varepsilon^2). \tag{C.13}$$

We write the Lagrangian now with the order parameter,

$$\begin{aligned} L(\mathbf{R}, \dot{\mathbf{R}}, \phi, \dot{\phi}, v_\parallel, v_\perp) = {} & \Big(mv_\parallel \mathbf{h}(\mathbf{R}) + mv_\perp \hat{\mathbf{n}}(\mathbf{R}, \phi) + \frac{e}{c}\varepsilon^{-1}\mathbf{A}(\mathbf{R} + \varepsilon\boldsymbol{\rho})\Big) \cdot (\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}) \\ & - \Big(\frac{m}{2}(v_\parallel^2 + v_\perp^2) + e\Phi(\mathbf{R} + \varepsilon\boldsymbol{\rho})\Big). \end{aligned} \tag{C.14}$$

Note, the ε^{-1} besides the vector potential since $A \sim B \sim \omega_c \sim \varepsilon^{-1}$. Further note that $\dot{\rho} \nsim \varepsilon\dot{\rho}$, which is demonstrated by

$$\begin{aligned} \dot{\boldsymbol{\rho}} &= \frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t} \\ &= (\dot{\mathbf{R}} \cdot \nabla)\boldsymbol{\rho} + \dot{\phi}\partial_\phi\boldsymbol{\rho} + \dot{v}_\perp\partial_{v_\perp}\boldsymbol{\rho} && \text{(C.15)} \\ &= \varepsilon(\dot{\mathbf{R}} \cdot \nabla)\boldsymbol{\rho} + \varepsilon^{-1}\dot{\phi}\varepsilon\rho_L\hat{\mathbf{n}} + \varepsilon\frac{\dot{v}_\perp}{v_\perp}\boldsymbol{\rho} && \text{(C.16)} \\ &= \dot{\phi}\rho_L\hat{\mathbf{n}} + \varepsilon\Big((\dot{\mathbf{R}} \cdot \nabla)\boldsymbol{\rho} + \frac{\dot{v}_\perp}{v_\perp}\boldsymbol{\rho}\Big). && \text{(C.17)} \end{aligned}$$

Note that the second term in the bracket might even be higher order, if the magnetic field is varying significantly.

Let's write expand the fields in the phase-space Lagrangian,

$$L(\mathbf{R}, \dot{\mathbf{R}}, \phi, \dot{\phi}, v_\parallel, v_\perp) = \tag{C.18}$$

$$\Big(mv_\parallel \mathbf{h}(\mathbf{R}) + mv_\perp \hat{\mathbf{n}}(\mathbf{R}, \phi) + \frac{e}{c}\varepsilon^{-1}\mathbf{A}(\mathbf{R}) + \frac{e}{c}\boldsymbol{\rho}\cdot\nabla\mathbf{A}(\mathbf{R})\Big)\cdot(\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}) - \Big(\frac{m}{2}(v_\parallel^2 + v_\perp^2) + e\Phi(\mathbf{R}) + e\varepsilon\boldsymbol{\rho}\cdot\nabla\Phi(\mathbf{R})\Big), \tag{C.19}$$

where for now we keep the $\dot{\boldsymbol{\rho}}$. Considering the first product to lowest order in ε, we have

$$\Big(mv_\parallel \mathbf{h}(\mathbf{R}) + mv_\perp \hat{\mathbf{n}}(\mathbf{R}, \phi) + \frac{e}{c}\varepsilon^{-1}\mathbf{A}(\mathbf{R}) + \frac{e}{c}\boldsymbol{\rho}\cdot\nabla\mathbf{A}(\mathbf{R})\Big)\cdot(\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}) \tag{C.20}$$

$$= \Big(mv_\parallel \mathbf{h} + mv_\perp \hat{\mathbf{n}} + \varepsilon^{-1}\frac{e}{c}\mathbf{A} + \frac{e}{c}\boldsymbol{\rho}\cdot\nabla\mathbf{A}\Big)\cdot\mathbf{R} + mv_\perp\dot{\phi}\rho_L + \varepsilon^{-1}\frac{e}{c}\mathbf{A}\cdot\dot{\boldsymbol{\rho}} + \frac{e}{c}\boldsymbol{\rho}\cdot\nabla\mathbf{A}\cdot\dot{\boldsymbol{\rho}} + \mathcal{O}(\varepsilon) \tag{C.21}$$

where we have used the expression (C.17) for $\dot{\boldsymbol{\rho}}$. Thus, the Lagrangian becomes

$$L(\mathbf{R}, \dot{\mathbf{R}}, \phi, \dot{\phi}, v_\parallel, v_\perp) + \mathcal{O}(\varepsilon) = \Big(mv_\parallel \mathbf{h} + mv_\perp \hat{\mathbf{n}} + \varepsilon^{-1}\frac{e}{c}\mathbf{A} + \frac{e}{c}\boldsymbol{\rho}\cdot\nabla\mathbf{A}\Big)\cdot\dot{\mathbf{R}} + mv_\perp\dot{\phi}\rho_L + \varepsilon^{-1}\frac{e}{c}\mathbf{A}\cdot\dot{\boldsymbol{\rho}} + \frac{e}{c}\boldsymbol{\rho}\cdot\nabla\mathbf{A}\cdot\dot{\boldsymbol{\rho}} - \Big(\frac{m}{2}(v_\parallel^2 + v_\perp^2) + e\Phi(\mathbf{R})\Big). \tag{C.22}$$

Now, we exploit the fact that adding a total time derivative to the Lagrangian does not change the equations of motion. First, consider

$$\begin{aligned}\nabla\mathbf{A}\cdot\dot{\mathbf{R}} &= \dot{\mathbf{R}}\times(\nabla\times\mathbf{A}) + \dot{\mathbf{R}}\cdot\nabla\mathbf{A}\\ &= \dot{\mathbf{R}}\times\mathbf{B} + \dot{\mathbf{R}}\cdot\nabla\mathbf{A}.\end{aligned} \tag{C.23}$$

which results in

$$\boldsymbol{\rho}\cdot\nabla\mathbf{A}\cdot\dot{\mathbf{R}} = \boldsymbol{\rho}\cdot\dot{\mathbf{R}}\times\mathbf{B} + \boldsymbol{\rho}\cdot\dot{\mathbf{R}}\cdot\nabla\mathbf{A}, \tag{C.24}$$

which is the last term in the first bracket in the Lagrangian. Next,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\rho}\cdot\mathbf{A}(\mathbf{R})) = \dot{\boldsymbol{\rho}}\cdot\mathbf{A} + \boldsymbol{\rho}\cdot(\dot{\mathbf{R}}\cdot\nabla)\mathbf{A}, \tag{C.25}$$

and together with (C.24) we have

$$\boldsymbol{\rho}\cdot\nabla\mathbf{A}\cdot\dot{\mathbf{R}} = \boldsymbol{\rho}\cdot\dot{\mathbf{R}}\times\mathbf{B} + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\rho}\cdot\mathbf{A}(\mathbf{R})) - \dot{\boldsymbol{\rho}}\cdot\mathbf{A}. \tag{C.26}$$

Note that the last term here cancels with the equivalent term in the Lagrangian (second term second line in (C.22). Also, we have

$$\begin{aligned}\frac{e}{c}\boldsymbol{\rho}\cdot(\dot{\mathbf{R}}\times\mathbf{B}) &= \frac{e}{c}\rho_L B\dot{\mathbf{B}}\cdot(\mathbf{h}\times\hat{\boldsymbol{\rho}}) \\ &= \frac{e}{c}\rho_L B\hat{\boldsymbol{\rho}}\cdot(\dot{\mathbf{R}}\times\mathbf{h}) \\ &= \frac{e}{c}\rho_L B\dot{\mathbf{R}}\cdot(\mathbf{h}\times\hat{\boldsymbol{\rho}}) \\ &= -mv_\perp\hat{\mathbf{n}}, \end{aligned}\tag{C.27}$$

and this term cancels with the second term in the first bracket in the Lagrangian. Thus, we have

$$\begin{aligned}L-\frac{e}{c}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\rho}\cdot\mathbf{A})+(\varepsilon) = \Big(mv_\parallel\mathbf{h}+\varepsilon^{-1}\frac{e}{c}\mathbf{A}\Big)\cdot\dot{\mathbf{R}}+mv_\perp\dot{\phi}\rho_L+\frac{e}{c}(\boldsymbol{\rho}\cdot\nabla)\mathbf{A}\cdot\boldsymbol{\rho} \\ -\Big(\frac{m}{2}(v_\parallel^2+v_\perp^2)+e\Phi\Big)\end{aligned}\tag{C.28}$$

Now, we resolve the last term on the first line with another total time derivative. Specifically, with

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\rho}\cdot\nabla\mathbf{A}\cdot\boldsymbol{\rho}) = \frac{1}{2}\Big(\dot{\boldsymbol{\rho}}\cdot\nabla\mathbf{A}\cdot\boldsymbol{\rho}+\boldsymbol{\rho}\cdot\nabla\mathbf{A}\cdot\dot{\boldsymbol{\rho}}\Big)+\frac{1}{2}\varepsilon\boldsymbol{\rho}\cdot\frac{\mathrm{d}\nabla\mathbf{A}}{\mathrm{d}t}\cdot\boldsymbol{\rho}.\tag{C.29}$$

Further, consider

$$\begin{aligned}(\boldsymbol{\rho}\times\dot{\boldsymbol{\rho}})\cdot\mathbf{B} &= (\boldsymbol{\rho}\times\dot{\boldsymbol{\rho}})\cdot(\nabla\times\mathbf{A}) \\ &= \varepsilon_{ijk}\rho_j\dot{\rho}_k\varepsilon_{imn}\partial_m A_n \\ &= (\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km})\rho_j\dot{\rho}_k\partial_m A_n \\ &= (\boldsymbol{\rho}\cdot\nabla)\mathbf{A}\cdot\dot{\boldsymbol{\rho}}-(\dot{\boldsymbol{\rho}}\cdot\nabla)\mathbf{A}\cdot\boldsymbol{\rho}.\end{aligned}\tag{C.30}$$

Let's look at the term of interest in the Lagrangian

$$\begin{aligned}\boldsymbol{\rho}\cdot\nabla\mathbf{A}\cdot\dot{\boldsymbol{\rho}} &= \boldsymbol{\rho}\cdot\nabla\mathbf{A}\cdot\dot{\boldsymbol{\rho}}-\frac{1}{2}\dot{\boldsymbol{\rho}}\cdot\nabla\mathbf{A}\cdot\boldsymbol{\rho}+\frac{1}{2}\dot{\boldsymbol{\rho}}\cdot\nabla\mathbf{A}\cdot\boldsymbol{\rho} \\ &= \frac{1}{2}\Big(\boldsymbol{\rho}\cdot\nabla\mathbf{A}\cdot\dot{\boldsymbol{\rho}}-\dot{\boldsymbol{\rho}}\cdot\nabla\mathbf{A}\cdot\boldsymbol{\rho}\Big)+\frac{1}{2}\Big(\boldsymbol{\rho}\cdot\nabla\mathbf{A}\cdot\dot{\boldsymbol{\rho}}+\dot{\boldsymbol{\rho}}\cdot\nabla\mathbf{A}\cdot\boldsymbol{\rho}\Big) \\ &= \frac{1}{2}(\boldsymbol{\rho}\times\dot{\boldsymbol{\rho}})\cdot\mathbf{B}-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\rho}\cdot\nabla\mathbf{A}\cdot\rho)+\mathcal{O}(\varepsilon).\end{aligned}\tag{C.31}$$

Again, the total time derivative does not affect the equations of motion and

$$\begin{aligned}(\boldsymbol{\rho}\times\dot{\boldsymbol{\rho}})\cdot\mathbf{B} &= (\rho_L\boldsymbol{\rho}\times\dot{\phi}\rho_L\hat{\mathbf{n}})\times\mathbf{B}+\mathcal{O}(\varepsilon) \\ &= -\dot{\phi}\rho_L^2 B+\mathcal{O}(\varepsilon) \\ &= -\frac{mcv_\perp}{e}\rho_L\dot{\phi}+\mathcal{O}(\varepsilon).\end{aligned}\tag{C.32}$$

So we have

$$L - \frac{e}{c}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\boldsymbol{\rho}\cdot\mathbf{A} - \frac{1}{2}\boldsymbol{\rho}\cdot\nabla\mathbf{A}\cdot\boldsymbol{\rho}\Big) + \mathcal{O}(\varepsilon) = \\ \Big(mv_\parallel + \frac{e}{c}\mathbf{A}\Big)\cdot\dot{\mathbf{R}} + \frac{1}{2}mv_\perp\rho_L\dot{\phi} - \Big(\frac{m}{2}(v_\parallel^2 + v_\perp^2) + e\Phi\Big) \quad \text{(C.33)}$$

And finally, we arrive at the guiding center Lagrangian,

$$L_{\mathrm{gc}}(\mathbf{R},\dot{\mathbf{R}},\phi,\dot{\phi},v_\parallel,v_\perp) = \Big(mv_\parallel\mathbf{h} + \frac{e}{c}\mathbf{A}\Big)\cdot\dot{\mathbf{R}} + \frac{m^2cv_\perp^2}{2eB}\dot{\phi} - \Big(\frac{m}{2}(v_\parallel^2 + v_\perp^2) + e\Phi\Big), \quad \text{(C.34)}$$

Note that there is another way of deriving this Lagrangian commonly presented in the literature. This other approach introduces a gyrophase average which integrates out the dependency on the gyrophase.

Appendix D

Classical Mechanics

D.1 Phase-space Lagrangian: simple harmonic oscillator

The Hamilton function of the simple oscillator in one dimension is given by

$$H(q,p) = \frac{p^2}{2m} + \frac{k}{2}q^2. \tag{D.1}$$

The corresponding phase-space Lagrangian is given by

$$L_{\mathrm{p}h}(q,\dot{q},p,\dot{p}) = p\dot{q} - H(q,p) \tag{D.2}$$

$$= p\dot{q} - \Big(\frac{p^2}{2m} + \frac{k}{2}q^2\Big). \tag{D.3}$$

Here, the generalized coordinates are q and p, and the corresponding velocities are $\dot{q}$ and $\dot{p}$. Furthermore, the Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{\mathrm{p}h}}{\partial \dot{q}} = \frac{\partial L_{\mathrm{p}h}}{\partial q} \tag{D.4}$$

$$\dot{p} = -kq = -\frac{\partial H}{\partial q}, \tag{D.5}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{\mathrm{p}h}}{\partial \dot{p}} = \frac{\partial L_{\mathrm{p}h}}{\partial \dot{p}} \tag{D.6}$$

$$0 = \dot{q} - \frac{p}{m} \Rightarrow \dot{q} = \frac{\partial H}{\partial p}, \tag{D.7}$$

which recovers Hamilton's equations.

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